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Application of Natural Decomposition Method for Solution of Fractional Black-Scholes Equation

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Abstract

The Black-Scholes equation is a partial differential equation that can model the European call option price problem. This equation can be of the order of natural numbers or fractional. The aim of this paper is to find a solution to the fractional order Black-Scholes partial differential equation. The method used to find solutions to these equations is the Natural decomposition method. Two numerical examples are presented in this paper. The results show that the Natural decomposition method is effective and easy to use to solve the fractional Black-Scholes equation.

Keywords: Natural transformation, adomian decomposition method, Black-Scholes equation, Caputo fractional derivative, option price

1. Introduction

In the financial and economic sector, investment activities have developed quite rapidly in recent years. This is indicated by the increasing variety of financial derivative products being developed as investment alternatives. Financial derivatives are investment products that are derivatives of a financial asset, so their value depends on those assets. This product is useful for minimizing the risk of loss, hedging or speculation. Some examples of financial derivative products include forward contracts, futures and options (Hull, 2012). Fisher Black and Myron Scholes (1973) first introduced the Black-Scholes partial differential equation which is used to calculate the price of European type call and put options, where the underlying financial asset is the stock price without dividend payments. Denoted $c - c(s,t)$ is the price of the European call option at the asset price S and time t. Let σ be the volatility of the asset price, E is the exercise price, T is the maturity time or date and r is the risk-free interest rate. The Black-Scholes equation and the boundary conditions for pricing European call options are (Wilmott et al., 1995; Gulkac, 2010):

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0, \tag{1}$$

with $c(0,t)$, $c(s,t) \sim s$ because $s \to \infty$ and $c(s,t) - \max\{S - E, 0\}$. Equation (1) resembles the diffusion equation with more parameters. Equation (1) can be simplified through the conversion as follows

$$S = Ee^x, \quad t = T - \frac{2\tau}{\sigma^2} \text{and} \quad C(S,t) = Ev(x,\tau)$$

Thus obtained partial differential equations

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv \tag{2}$$

Where $K = \frac{2r}{\sigma^2}$ and initial conditions become $v(x,0)x = \max\ (e^x - 1,0\}$.

Various methods were developed to solve the Black-Scholes partial differential equation, such as the homotopy pertubation method (Gulkac, 2010), the finite difference method (Cen & Le, 2011), the homotopy analysis method, the variation iteration method (Allahviranloo & Behzadi, 2013), the transformation method projected differential (Edeki et al., 2015) and the Adomian decomposition method (Edeki, et al., 2015; Gonzalez et al., 2017). Several models were developed to weaken the rigid assumptions of the classic Black-Scholes equation, such as models with transaction costs, models with volatility and interest rates in the form of non-constant functions, and Black-Scholes models with fractional time. Fractional calculus deals with derivatives and integrals of fractional order. Apart from the fields of finance and economics, fractional calculus is also widely applied to the fields of physics, engineering, chemistry, biology and the environment (David et al., 2011; Rusyaman et al., 2018; Sumiati et al., 2018; Sun et al., 2018). The method used to solve the fractional Black-Scholes equation is almost similar to the method used to solve the classical Black-Scholes equation. For example, the homotopy pertubation method (Kumar et al., 2012; Ouafoudi & Gao, 2018), the homotopy analysis method (Kumar et al., 2014), the variation iteration method (Ghandehari & Ranjbar, 2014a; Eshaghi et al., 2017), the method finite difference (Akrami & Erjaee, 2016), projected differential transformation method (Edeki et al., 2017) and Adomian decomposition method (Ghandehari & Ranjbar, 2014b; Yavuz & Ozdemir, 2018).

The Adomian decomposition method is an effective and useful iterative method for finding analytical solutions without linearization, perturbation, transformation or discretization (Adomian, 1988). This decomposition method is capable of solving ordinary or partial differential equations with integer or fractional order, with initial or limit value problems, with constant or variable coefficients, linear or nonlinear, and homogeneous or nonhomogeneous (Duan et al., 2012; Al awawdah, 2016). The Adomian decomposition method can be combined with integral transformations, such as Laplace (Khuri, 2001), Sumudu (Khan et al., 2008) and Natural (Baskonus et al., 2014).

The Natural decomposition method is capable of solving ordinary or partial differential equations, both integer and fractional. In this paper, the Natural decomposition method is used to solve the fractional order Black-Scholes partial differential equation. Previous studies have shown that the Natural decomposition method is very useful and easy to use to solve the Fokker-Planck, Schrödinger, Kelin-Gorden equations (Abdel-Randy et al., 2015), telegraph (Eltayeb et al., 2019) and diffusion (Shah et al., 2019) is of fractional order.

2. Basic theory

This section presents the basic theories and properties that support and support this research, such as Natural transformations, Mittag Leffler functions and fractional derivatives.

Definition 1 (Belgacem & Silambarasan, 2012) The natural transformation of the function $f(t) \in R^2$is expressed as the following integral equation

$$\boldsymbol{N}[f(t)] = R(s,u) = \int_0^\infty e^{-st} f(ut)\,dt = \lim_{b\to\infty}\int_0^b e^{-t} f(ut)\,dt\,, Re(s) > 0, u \in (-\tau_1, \tau_2)$$

where the function f is defined as a set

$$A = \left\{ f(t) | \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}}, \text{if } t \in (-1)^j \times [0,\infty) \right\}. \tag{3}$$

For $u = 1$, Definition 1 is equivalent to the Laplace transformation (Shiff, 1999), while for $s = 1$, Definition 1 is equivalent to the Sumudu transformation (Belgacem & Karaballi, 2006), respectively it can be rewritten as follows

$$\boldsymbol{L}[f(t)] = R(s,1) = \int_0^\infty e^{-st} f(t)\,dt\,, s > 0$$

$$\boldsymbol{S}[f(t)] = R(1,u) = \int_0^\infty e^{-t} f(ut)\,dt, u > 0.$$

According to Definition 1, for $f(t) = t^n$where $t \geq 0$and *n* are nonnegative integers, the Natural transformation of *f* is

$$\boldsymbol{N}[t^n] = n!\frac{u^n}{s^{n+1}}. \tag{4}$$

If $\alpha \in \boldsymbol{R}$, then equation (4) can be written

$$\boldsymbol{N}[t^\alpha] = \Gamma(\alpha+1)\frac{u^\alpha}{s^{\alpha+1}}. \tag{5}$$

where $\Gamma(x)$ is the Gamma function. Also, according to Definition 1, a Natural transformation of an *n order derivative* can be written

$$\boldsymbol{N}\left[f^{(n)}(t)\right] = \frac{s^n}{u^n}R(s,u) - \sum_{k=0}^{n-1}\frac{s^{n-k-1}}{u^{n-k}} f^{(k)}(0).$$

Definition 2 (Akgül, 2018) The basic Mittag-Leffler function is denoted by $E_\alpha(z)$ where $\alpha \in \boldsymbol{R}$, $Re(\alpha) > 0$and $z \in \boldsymbol{C}$, are defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}.$$

Definition 3 (Podlubny, et al., 2002) The Riemann-Liouville fractional derivative of the function *f* with respect to *t* with order $\alpha > 0$is defined as

$$ {}_{a}^{c}D_{t}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-y)^{n-\alpha-1}f(y)\,dy, n-1<\alpha\le n. $$

Definition 4 (Podlubny, et al., 2002) The fractional derivative of the Caputo function *f* with respect to *t* with order $\alpha > 0$is defined as

$$ {}_{a}^{C}D_{t}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-y)^{n-\alpha-1}f^{(n)}(y)\,dy, n-1<\alpha\le n. $$

Definition 5 (Abdel-Randy et al., 2015) The natural transformation of the Caputo fractional derivative is defined as

$$ \boldsymbol{N}\left[{}^{C}D_{x}^{\alpha}f(t)\right] = \frac{s^{\alpha}G(u)}{u^{\alpha}} - \sum_{k=0}^{n-1}\frac{s^{\alpha-k-1}}{u^{\alpha-k}}f^{(k)}(0), n-1<\alpha\le n. $$

3. Natural Decomposition Method

The Adomian decomposition method assumes that the solution is decomposed into an infinite series, the nonlinear form (if any) is decomposed into an Adomian polynomial and an iterative algorithm is constructed to determine the solution recursively. The numerical scheme of the Natural transformation based on the Adomian decomposition method applies the Natural transformation and its inverse to differential equations (Baskonus et al., 2014).

Given the fractional partial differential equation as follows

$$ D_t^{\alpha}w(x,t) + Nw(x,t) + Rw(x,t) = g(x,t) \text{ and initial conditions } w(x,0) = f(x) \tag{6} $$

where $D_t^{\alpha}$is the Caputo fractional derivative operator with $0 < \alpha \le 1$, *N* is the nonlinear operator, *R* is the linear operator, *g* is the function that shows the nonhomogeneity of the differential equation and *w* is the function to be determined. Equation (6) can be rewritten $D_t^{\alpha}w(x,t)$ as the subject

$$ D_t^{\alpha}w(x,t) = g(x,t) - Nw(x,t) - Rw(x,t)\,. \tag{7} $$

Use the Natural transformation in equation (7) to obtain

$$ w(x,s,u) = \frac{w(x,0)}{s} + \frac{u^{\alpha}}{s^{\alpha}}\boldsymbol{N}[g(x,t)] - \frac{u^{\alpha}}{s^{\alpha}}\boldsymbol{N}[Nw(x,t)] - \frac{u^{\alpha}}{s^{\alpha}}\boldsymbol{N}[Rw(x,t)] \tag{8} $$

Use the inverse Natural transformation in equation (8)

$$ w(x,t) = w(x,0) + \boldsymbol{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\boldsymbol{N}[g(x,t)]\right] - \boldsymbol{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\boldsymbol{N}[Nw(x,t)] + \frac{u^{\alpha}}{s^{\alpha}}\boldsymbol{N}[Rw(x,t)]\right] \tag{9} $$

The Adomian decomposition method assumes that the function *w* can be decomposed into an infinite series

$$ w = \sum_{n=0}^{\infty} w_n \tag{10} $$

which $w_n$can be determined recursively. This method also assumes that the nonlinear operator *Nw* can be decomposed by an infinite polynomial series

$$ Nw = \sum_{n=0}^{\infty} A_n \tag{11} $$

where $A_n = A_n(w_0, w_1, w_2, \ldots, w_n)$ is the defined Adomian polynomial

$$ A_n(w_0, w_1, w_2, \ldots, w_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{n}\lambda^k w_k\right)\right]_{\lambda=0} \quad ; n = 0,1,2,\ldots $$

where λ is the parameter, the adomian $A_{n\ polynomial}$ can be described as follows

$$ A_0 = \frac{1}{0!}\frac{d^0}{d\lambda^0}\left[N\left(\sum_{k=0}^{0}\lambda^k w_k\right)\right]_{\lambda=0} = N(w_0), $$

$$ A_1 = \frac{1}{1!}\frac{d^1}{d\lambda^1}\left[N\left(\sum_{k=0}^{1}\lambda^k w_k\right)\right]_{\lambda=0} = w_1 N(w_0), $$

$$A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}\left[N\left(\sum_{k=0}^{2}\lambda^k w_k\right)\right]_{\lambda=0} = \frac{w_1^2}{2!}N''^{(w_0)} + w_2 N'(w_0),$$

Substitute initial conditions, equations (10) and (11) into equation (9)

$$\sum_{n=0}^{\infty} w_n = f(x) + \boldsymbol{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\boldsymbol{N}[g(x,t)]\right] - \boldsymbol{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\boldsymbol{N}\left[\sum_{n=0}^{\infty} A_n\right] + \frac{u^\alpha}{s^\alpha}\boldsymbol{N}\left[R\sum_{n=0}^{\infty} w_n\right]\right] \quad (12)$$

If both sides of equation (12) are described, then they are obtained successively

$$w_0 = f(x) + \mathbf{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbf{N}[g(x,t)]\right]$$

$$w_1 = -\mathbf{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbf{N}[A_0] + \frac{u^\alpha}{s^\alpha}\mathbf{N}[Rw_0]\right]$$

$$w_2 = -\mathbf{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbf{N}[A_1] + \frac{u^\alpha}{s^\alpha}\mathbf{N}[Rw_1]\right]$$

$$w_3 = -\mathbf{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbf{N}[A_3] + \frac{u^\alpha}{s^\alpha}\mathbf{N}[Rw_2]\right]$$

Thus, in general, the recursive relations obtained from the solution of the fractional partial differential equation (6) are as follows

$$w_0 = f(x) + \mathbf{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbf{N}[g(x,t)]\right],$$

$$w_{n+1} = -\mathbf{S}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbf{N}[A_n] + \frac{u^\alpha}{s^\alpha}\mathbf{N}[Rw_n]\right], \qquad n = 0,1,2,\dots \quad (13)$$

Therefore, the approximate solution of equation (13) is

$$w \approx \sum_{n=0}^{k} w_n, \text{ where } \lim_{k\to\infty}\sum_{n=0}^{k} w_n = w. \quad (14)$$

The main advantage of this technique is that the solution can be expressed as an infinite series that converges quickly to the exact solution (Sahoo & Patra, 2021; Rawashdeh & Maitama, 2017).

4. Numerical Example

Two numerical examples of applying the Natural decomposition method to solve fractional Black-Scholes partial differential equations are presented in this section.

Example 1. Based on equation (3), the fractional Black-Scholes partial differential equation is given as follows

$$\frac{\partial^\alpha v}{\partial t^\alpha} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv, \quad (15)$$

with initial conditions $v(x,0) = max\{e^x - 1, 0\}$.

The fractional Black-Scholes equation (15) can be solved using the Natural decomposition method as follow

$$v_0 = max\{e^x - 1, 0\},$$

$$v_{n+1} = \boldsymbol{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\boldsymbol{N}\left[\frac{\partial^2 v_n}{\partial x^2} + (k-1)\frac{\partial v_n}{\partial x} - kv_n\right]\right], n = 0,1,2,\dots \quad (16)$$

If the recursive solution is parsed then it is obtained

$$v_1 = \mathbf{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbf{N}\left[\frac{\partial^2 v_n}{\partial x^2} + (k-1)\frac{\partial v_n}{\partial x} - kv_0\right]\right],$$

$$= \mathbf{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbf{N}[kmax\{e^x, 0\} - kmax\{e^x - 1, 0\}]\right],$$

$$= \mathbf{N}^{-1}\left[\frac{ku^\alpha max\{e^x, 0\} - ku^\alpha \boldsymbol{S} max\{e^x - 1, 0\}}{s^{\alpha+1}}\right],$$

$$= \frac{t^{\alpha}}{\Gamma(\alpha+1)}(kmax\{e^x, 0\} - kmax\{e^x - 1{,}0\})$$

Because $\frac{\partial v_1}{\partial x} = \frac{t^{\alpha}}{\Gamma(\alpha+1)}(k\,max\{e^x, 0\} - k\,max\{e^x, 0\}) = 0$, so

$$v_2 = \mathbf{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbf{N}\left[\frac{\partial^2 v_1}{\partial x^2} + (k-1)\frac{\partial v_1}{\partial x} - kv_1\right]\right],$$

$$= \mathbf{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbf{N}\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)}(-k^2\ max\{e^x, 0\} + k^2 max\{e^x - 1{,}0\})\right]\right],$$

$$= \mathbf{N}^{-1}\left[\frac{-k^2u^{2\alpha}\,max\{e^x,0\} + k^2u^{2\alpha}max\{e^x - 1{,}0\}}{s^{2\alpha+1}}\right]$$

$$= \frac{t^{\alpha}}{\Gamma(\alpha+1)}(-k^2\,max\{e^x, 0\} + k^2 max\{e^x - 1{,}0\})$$

$$v_3 = \mathbf{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbf{N}\left[\frac{\partial^2 v_2}{\partial x^2} + (k-1)\frac{\partial v_3}{\partial x} - kv_3\right]\right],$$

$$= \mathbf{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbf{N}\left[\frac{t^{\alpha}}{\Gamma(2\alpha+1)}(-k^3\ max\{e^x, 0\} + k^3 max\{e^x - 1{,}0\})\right]\right]$$

$$= \mathbf{N}^{-1}\left[\frac{-k^3u^{3\alpha}\,max\{e^x,0\} + k^3u^{3\alpha}max\{e^x - 1{,}0\}}{s^{2\alpha+1}}\right]$$

$$= \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}(-k^3\,max\{e^x, 0\} + k^3 max\{e^x - 1{,}0\})$$

Based on these iterations, the solution of the fractional Black-Scholes equation (15) can be expressed as a convergent infinite series as follows

$$v\,v(x,\tau) = \sum_{n=0}^{\infty} v_n(x,\tau) = \max\{e^x - 1, 0\}E_{\alpha}(-k\tau^{\alpha}) + \max\{e^x, 0\}(1 - E_{\alpha}(-k\tau^{\alpha})) \quad (17)$$

For $\alpha = 1$, then obtained

$$v(x,t) = max\{e^x - 1{,}0\}\,e^{-kt} + max\{e^x, 0\}\,(1 - e^{-kt})$$

The solution above is equivalent to the exact solution of the Black-Scholes partial differential equation (Ghandehari & Ranjbar, 2014b; Yavuz & Ozdemir, 2018).

Example 2. Given a generalized fractional Black-Scholes partial differential equation as follows (Cen & Le, 2011; Kumar et al., 2012; Yavuz & Ozdemir, 2018)

$$\frac{\partial^{\alpha} v}{\partial t^{\alpha}} + 0.08(2 + sin\,x)^2x^2\frac{\partial^2 v}{\partial x^2} + 0.06x\frac{\partial v}{\partial x} - 0.06v = 0 \quad (18)$$

with initial conditions $v(x, 0) = max\{x - 25e^{-0.06}, 0\}$.

The solution to the fractional Black-Scholes equation (18) using the Natural decomposition method is as follows

$$v_0 = v\,(x, 0) = max\{x - 25e^{-0.06}, 0\},$$

$$v_{n+1} = \mathbf{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbf{N}\left[-0.08(2 + sinx)^2x^2\frac{\partial^2 v_n}{\partial x^2} - 0.06x\frac{\partial v_n}{\partial x} + 0.06v_n\right]\right],\ \ n = 0,1,2,..., \quad (19)$$

If the recursive solution is parsed then it is obtained

$$v_1 = \mathbf{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbf{N}\left[-0.08(2 + sinx)^2x^2\frac{\partial^2 v_n}{\partial x^2} - 0.06x\frac{\partial v_n}{\partial x} + 0.06v_n\right]\right]$$

$$= \mathbf{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbf{N}\,[-0.06x + 0.06\,max\{x - 25e^{-0.06}, 0\}]\right]$$

$$= \mathbf{N}^{-1}\left[\frac{-0.06u^{\alpha}ax\{x - 25e^{-0.06}, 0\}}{s^{\alpha+1}}\right]$$

$$= \frac{t^{\alpha}}{\Gamma(\alpha+1)}(-0.06x + 0.06\,max\{x - 25e^{-0.06}, 0\})$$

Because $\frac{\partial v_1}{\partial x} = \frac{t^{\alpha}}{\Gamma(\alpha+1)}(-0.06 + 0.06) = 0$, so

$$v_2 = \mathbf{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbf{N}\left[-0.08(2+sinx)^2x^2\frac{\partial^2 v_1}{\partial x^2} - 0.06x\frac{\partial v_1}{\partial x} + 0.06v_1\right]\right]$$

$$= \mathbf{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbf{N}\left[\frac{t^\alpha}{\Gamma(\alpha+1)}\left(-(0.06)^2x + (0.06\,)^2 max\{x - 25e^{-0.06}, 0\}\right)\right]\right]$$

$$= \mathbf{N}^{-1}\left[\frac{-(0.06)^2u^{2\alpha}x + (0.06)^2u^{2\alpha}max\{x - 25e^{-0.06}, 0\}}{s^{2\alpha+1}}\right]$$

$$= \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\left(-(0.06)^2x + (0.06)^2\, max\{x - 25e^{-0.06}, 0\}\right)$$

$$v_3 = \mathbf{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbf{N}\left[-0.08(2+sinx)^2x^2\frac{\partial^2 v_2}{\partial x^2} - 0.06x\frac{\partial v_2}{\partial x} + 0.06v_2\right]\right]$$

$$= \mathbf{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbf{N}\left[\frac{t^{2\alpha}}{\Gamma(\alpha+1)}\left(-(0.06)^3x + (0.06\,)^3 max\{x - 25e^{-0.06}, 0\}\right)\right]\right]$$

$$= \mathbf{N}^{-1}\left[\frac{-(0.06)^3u^{3\alpha}x + (0.06)^3u^{3\alpha}max\{x - 25e^{-0.06}, 0\}}{s^{3\alpha+1}}\right]$$

$$= \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\left(-(0.06)^3x + (0.06)^3\, max\{x - 25e^{-0.06}, 0\}\right)$$

Therefor the solution of the generalization of the fractional Black-Scholes equation (18) is written as equation (20).

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t) = max\{\,x - 25e^{-0.06}, 0\}E_\alpha(0.06t^\alpha) + x(1 - E_\alpha(0.06t^\alpha)) \qquad (20)$$

For $\alpha = 1$, then obtained

$$v(x,t) = max\{\,x - 25e^{-0.06}, 0\}e^{0.06t} + x(1 - e^{0.06t})$$

5. Conclusion

The Black-Scholes equation is a very well-known model for pricing options. In this paper, a combination of the Adomian decomposition method and the Natural transformation is used to solve the fractional order Black-Scholes equation, where the fractional derivative used is the Caputo derivative. Based on the two numerical examples given, the Natural decomposition method has the advantage that the solution can be expressed as an infinite series that converges rapidly to its exact solution. Thus, the Natural decomposition method is very effective, useful and easy to use to solve fractional Black-Scholes partial differential equations with boundary conditions for European option pricing problems.

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