Available online at http://www.iorajournal.org/index.php/ijgor/index

International Journal of Global Operations Research

e-ISSN: 2722-1016
p-ISSN: 2723-1739

Vol. 4, No. 2, pp. 59-67, 2023

Controlling Water Quality of a Firm's Hydropower

Najeeb Al-Matar[1*], Lotfi Tadj[2]

[1] *Department of Business Administration, Al-Baha University, Alaqiq, Saudi Arabia*
[2] *Department of Industrial Engineering, Alfaisal University, Riyadh, Saudi Arabia*

**Corresponding author email: nalmatar@bu.edu.sa*

Abstract

The problem considered in this paper concerns a manufacturing firm located at a lake. The firm uses the lake's water in its production process. However, the production process affects the quality of the water. The manufacturing firm will only survive if the natural resource it uses, namely water in this case, remains available for the long term. Sustainability can be achieved through green manufacturing practices. The problem is dynamic, and optimal control theory, namely model predictive control, is used to determine the optimal production rate and the optimal rate of expenditure for pollution control. A numerical example illustrates the results obtained.

Keywords: Sustainability, green manufacturing, water quality, optimal control, tracking, pollution control.

1. Introduction

There is no doubt that green manufacturing has widespread environmental benefits. In green manufacturing, fewer natural resources are used, and pollution and waste are reduced. This is accomplished by adopting practices that influence the product design, process design, and operational principles. Impacts on the environment are lessened.

Clean and green manufacturing, or sustainable manufacturing, concerns in particular systems that use renewable energy. Fewer natural resources are used, and lean technology equipment of all kinds are utilized to lower pollution, minimize waste, and reduce emissions. Sustainable manufacturing can be applied in all manufacturing sectors that minimize waste and pollution, enables economic progress and conserve resources. It lead to environmental protection.

The set of problems that relate to governing economic activity in order to promotes human well-being, sustainability, and justice is known as ecological economics. Optimal control theory has been used extensively to solve a variety of such problems. For example, (Highfill and McAsey, 2001) use optimal control theory to choose the appropriate levels of the two disposal techniques: landfilling vs recycling. They show that once recycling begins, it will increase.

Eutrophication is the process through which a body of water, such as a lake, becomes excessively covered with plant and algae. This is usually due to increased nutrients in the water from man-made sources. (Nævdal, 2001) analyzes the optimal regulation of eutrophying of bodies of water in the presence of threshold effects. He shows that fertilizer use should decrease with time until the lake is able to clean itself. At that point, fertilizer use is allowed to increase.

Pollution is the introduction of harmful materials into the environment. It has catastrophic effects on the earth. (Kawaguchi, 2003) introduces a pollution accumulation model. He shows that social pollution control policy should aim at the optimal stationary level of pollution, not only the growth rate or the flow. Pollution is also the subject of (Hartl et al., 2006). They consider a set of production plants which use water from some lake that they pollute during the production process. They derive the optimal amounts of water to be taken from the lake or to be pumped back from a waste water reservoir.

Water resources can be used to generate electricity. They can also have other uses such as industry, irrigation, etc. da Silva and (de Souza, 2008) present a dynamic model to analyze the tradeoff between different uses of water. They establish that the price of water for non-energy uses should be twice the price of the energy goods, indicating the necessity to substitute other sources of energy for hydroelectric power.

Optimal control problems are usually formulated with a single objective function and one or more differential equations, known as state equations. There is a single decision maker that tries to optimize the objective function. When there is more than one decision maker, each having an objective function they try to optimize, subject to a set of state equations, then an extension of optimal control theory, known as the theory of differential games is used.

(Brock and Dechert, 2008) build a simple model of two spatially connected ecosystems. They formulate a dynamic game for the case where management authorities cannot coordinate. Use of games to tackle pollution problems have also been considered by (Kossioris et al., 2008) and (Tapiero, 2005a, 2005b, 2009). See (Jørgensen et al., 2010) for a survey of literature which utilizes games to formulate and analyze multi-objective problems in the economics and management of pollution. For a review of differential games applications to management science and operations research problems, see (Feichtinger et al., 1983).

The problem we are interested in in this paper is that of pollution control. It was first described by (Sethi, 1977). It concerns a manufacturing firm located on a lake. The firm uses hydropower, or hydroenergy, a form of renewable energy that uses the water stored in the lake to create electricity in hydropower plants. It is assumed that the production process pollutes the lake. The pollution, in turn, increases the firm's production costs. The control problem is to determine the optimal production levels.

This problem has also been considered by a few other authors. (Feichtinger, 1982) views Sethi's model as a differential game between a governmental agency for cleaning the environment and a firm polluting the environment by producing an output. The governmental cleaning agency aims at providing a high level of environment quality taking into consideration the expenditures for its cleaning. (Hartl, 1982, 1983) considers a general nonlinear model of which Sethi's model is a special case. Hartl uses stability analysis to study the qualitative properties of the optimal policy. Other models related to Sethi's are that of (Clarke, 1987) who takes into account externalities in production and (Feichtinger, 1988) who incorporates business cycles.

We integrate the following features into Sethi's model. First, some of the parameters that were constant in Sethi's model are dynamic in our model. These parameters are the self-cleaning ability rate of the water and the pollutivity of a unit production. Second, we take into account the inventory-production process. It seems natural to consider this process since production affects the quality of the lake. Third, we assume that items produced are subject to deterioration, and the deterioration rate is dynamic. Fourth, the system is assumed to be of the tracking-type, as is customary in many engineering management problems. Finally, instead of looking for the optimal solution at the beginning of the planning horizon, our solution procedure optimizes the system over small successive intervals, known as prediction intervals.

The formulation of the model is presented in Section 2 and the solution to the model is presented in Section 3. A numerical example is treated in Section 4 and the paper is summarized in Section 5.

2. Model Formulation

Consider a manufacturing firm on a lake. The firm uses the water from the lake for its production process. The production process in turn pollutes the lake. Let $\boldsymbol{H}$ denote the length of the planning horizon, and for any $\boldsymbol{t_0} \in [\boldsymbol{0}, \boldsymbol{H}]$, let $[\boldsymbol{t_0}, \boldsymbol{t_0} + \boldsymbol{T}]$ represent the prediction horizon. Here $\boldsymbol{T} > \boldsymbol{0}$ is much smaller than $\boldsymbol{H}$. Using Sethi's notation, we introduce the state variable $\boldsymbol{Q}(\boldsymbol{t})$ which represents the quality of the lake at time $\boldsymbol{t}$, and the control variables $\boldsymbol{P}(\boldsymbol{t})$ and $\boldsymbol{u}(\boldsymbol{t})$ which represents the production rate at time $\boldsymbol{t}$ and the rate of expenditure for pollution control at time $\boldsymbol{t}$, respectively. Letting $\boldsymbol{\lambda}(\boldsymbol{t})$ and $\boldsymbol{\alpha}(\boldsymbol{t})$ denote the self-cleaning ability rate of the water at time $\boldsymbol{t}$ and the pollutivity of a unit production at time $\boldsymbol{t}$, respectively, the dynamics of the lake quality are governed by the following state equation, see (Sethi, 1977):

$$\frac{d}{dt}Q(t) = -[\lambda(t) + u(t)]Q(t) + \alpha(t)P(t),\ Q(0) = Q_0. \tag{1}$$

The initial value $\boldsymbol{Q_0}$ is known. The exogenous variables $\boldsymbol{\lambda}(\boldsymbol{t})$ and $\boldsymbol{\alpha}(\boldsymbol{t})$ are constant in Sethi's model, while we assume that they are dynamic in our case for more generality.

To incorporate the activities of the firm in the model, we introduce the second state variable, $\boldsymbol{I}(\boldsymbol{t})$, which represents the inventory level at time $\boldsymbol{t}$. The initial inventory level $\boldsymbol{I_0}$ is known. The items are assumed to deteriorate while on the shelf at a dynamic rate $\boldsymbol{\theta}(\boldsymbol{t})$. The evolution of the inventory level occurs according to the well-known differential equation

$$\frac{d}{dt}I(t) = P(t) - D(t) - \theta(t)I(t),\ I(0) = I_0. \tag{2}$$

Following (Sethi, 2019), we assume a system of the tracking type where the firm sets targets for the state variables: a target inventory level at time t denoted by $\hat{I}(t)$, and a target quality of the lake at time t denoted by $\hat{Q}(t)$. The corresponding target control variables are a target production rate at time t denoted $\hat{P}(t)$ and a target pollutivity of a unit production at time t denoted $\hat{u}(t)$. Because the target variables must satisfy the state equation (1) and equation (2), we can easily show that the target control variables are given by

$$\hat{P}(t) = \frac{d}{dt}\hat{I}(t) + D(t) + \theta(t)\hat{I}(t), \tag{3}$$

and

$$\hat{u}(t) = \frac{1}{\hat{Q}(t)}\left[-\frac{d}{dt}\hat{Q}(t) - \lambda(t)\hat{Q}(t) + \alpha(t)\hat{P}(t)\right]. \tag{4}$$

The firm aims at having each variable, either state or control, as close as possible to its respective target during the prediction interval. Thus, the following penalties are incurred for each deviation:

p_1 : penalty when $I(t)$ deviates from $\hat{I}(t)$;
p_2 : penalty when $Q(t)$ deviates from $\hat{Q}(t)$;
q_1 : penalty when $P(t)$ deviates from $\hat{P}(t)$;
q_2 : penalty when $u(t)$ deviates from $\hat{u}(t)$;
c_1 : final inventory state penalty;
c_2 : final quality state penalty.

The following objective function is selected to be minimized:

$$\begin{aligned} J = \; & \frac{1}{2}\int_{t_0}^{t_0+T} \{p_1 \Delta I(t)^2 + p_2 \Delta Q(t)^2 + q_1 \Delta P(t)^2 + q_2 \Delta u(t)^2\} dt \\ & + \frac{c_1}{2}\Delta I(t_0+T)^2 + \frac{c_2}{2}\Delta Q(t_0+T)^2, \end{aligned} \tag{5}$$

where the shift operator Δ is defined by $\Delta f(t) = f(t) - \hat{f}(t)$. Finally, the state equation (1) and equation (2) are rewritten under the following, more convenient forms:

$$\frac{d}{dt}\Delta I(t) = \Delta P(t) - \theta(t)\Delta I(t), \tag{6}$$

and

$$\frac{d}{dt}\Delta Q(t) = -[\lambda(t) + \hat{u}(t)]\Delta Q(t) - Q(t)\Delta u(t) + \alpha(t)\Delta P(t). \tag{7}$$

3. Model Solution

We need to solve the problem of finding the optimal control variables $\boldsymbol{P(t)}$ and $\boldsymbol{u(t)}$ by minimizing the objective function equation (5) subject to the state equation (6) and equation (7). The integral in equation (5) is calculated using the trapezoid formula of numerical analysis. The prediction horizon $[\boldsymbol{t_0}, \boldsymbol{t_0} + \boldsymbol{T}]$ is divided into $\boldsymbol{m}$ subintervals of equal length $\boldsymbol{h} = \boldsymbol{T}/\boldsymbol{m}$. For convenience, we will write $\boldsymbol{t}$ instead of $\boldsymbol{t_0}$ in the rest of the paper. We will use a matrix notation by introducing the following $\boldsymbol{m} \times \boldsymbol{1}$ vectors.

$$\begin{aligned} U(t) &= [\Delta P(t), \Delta P(t+h), \Delta P(t+2h), \cdots, \Delta P(t+(m-1)h)]^T, \\ V(t) &= [\Delta u(t), \Delta u(t+h), \Delta u(t+2h), \cdots, \Delta u(t+(m-1)h)]^T, \\ G_1(t) &= [x_1(t), 0, \cdots, 0]^T, \\ G_2(t) &= [x_2(t), 0, \cdots, 0]^T, \end{aligned}$$

and $m \times m$ diagonal matrices

$$Q_1(t) = \text{diag}\left[x_3(t), \frac{hq_1}{2}, \cdots, \frac{hq_1}{2}\right],$$

$$Q_2(t) = \text{diag}\left[x_4(t), \frac{hq_2}{2}, \cdots, \frac{hq_2}{2}\right],$$

and

$$Q_{12}(t) = \begin{bmatrix} x_5(t) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

where

$$x_1(t) = \left\{h^2\delta p_1 - h^3\beta p_1\theta(t) + \frac{h^2 m}{2}p_1 - \frac{h^3 m^2 p_1}{2}\theta(t) + c_1 hm - c_1 h^2 m^2\theta(t)\right\}\Delta I(t)$$
$$+\left(\left[\frac{h^2 m}{2}p_2 + c_2 hm\right]\{1 - mh[\lambda(t) + \hat{u}(t)]\} + h^2 p_2\{\delta - \beta h[\lambda(t) + \hat{u}(t)]\}\right)\alpha(t)\Delta Q(t),$$
$$x_2(t) = \left(p_2 h^2\{\delta - \beta h[\lambda(t) + \hat{u}(t)]\} + \left[\frac{p_2 h^2 m}{2} + c_2 hm\right]\{1 - mh[\lambda(t) + \hat{u}(t)]\}\right)Q(t)\Delta Q(t),$$
$$x_3(t) = \frac{hq_1}{4} + \frac{h^3\beta}{2}[p_1 + p_2\alpha(t)^2] + \frac{h^3 m^2}{4}[p_1 + p_2\alpha(t)^2] + \frac{c_1 h^2 m^2}{2} + \frac{c_2 h^2 m^2}{2}\alpha(t)^2,$$
$$x_4(t) = \frac{hq_2}{4} + \left(\frac{p_2 h^3 m^2}{4} + \frac{\beta p_2 h^3}{2} + \frac{c_2 h^2 m^2}{2}\right)Q(t)^2,$$

and

$$x_5(t) = \left(\beta p_2 h^3 + \frac{p_2 h^3 m^2}{2} + c_2 h^2 m^2\right)\alpha(t)Q(t)$$

with $\delta = m(m-1)/2$ and $\beta = m(m-1)(2m-1)/6$. Using this notation, the objective function equation (5) is rewritten as

$$J = x_0(t) + G_1^\top(t)U(t) - G_2^\top(t)V(t) - U^\top(t)Q_{12}(t)V(t) + U^\top(t)Q_1(t)U(t) + V^\top(t)Q_2(t)V(t), \tag{8}$$

where $\boldsymbol{x_0(t)}$ is independent of the control variables and is given by

$$x_0(t) = \left(\frac{p_2 h}{2}\{(m-1) - 2\delta h[\lambda(t) + \hat{u}(t)] + \beta h^2[\lambda(t) + \hat{u}(t)]^2\} + \left[\frac{hp_2}{4} + \frac{c_2}{2}\right]\{1 - mh[\lambda(t) + \hat{u}(t)]\}^2 + \frac{hp_2}{4}\right)\Delta Q(t)^2$$
$$+ \left\{\frac{hmp_1}{2} + \frac{c_1}{2} - \left[\delta p_1^2 + \frac{p_1 h^2 m}{2} + c_1 hm\right]\theta(t) + \left[\frac{\beta p_1 h^3}{2} + \frac{p_1 h^3 m^2}{4} + \frac{c_1 h^2 m^2}{2}\right]\theta(t)^2\right\}\Delta I(t)^2.$$

The necessary optimality conditions

$$G_1(t) - Q_{12}(t)V(t) + 2Q_1(t)U(t) = 0, \tag{9}$$
$$G_2(t) + Q_{12}(t)U(t) - 2Q_2(t)V(t) = 0, \tag{10}$$

yield the optimal control variables

$$\Delta P(t) = \frac{x_2(t)x_5(t) - 2x_1(t)x_4(t)}{4x_3(t)x_4(t) - x_5(t)^2}, \tag{11}$$
$$\Delta u(t) = \frac{x_1(t)x_5(t) - 2x_2(t)x_3(t)}{4x_3(t)x_4(t) - x_5(t)^2}. \tag{12}$$

Substituting equation (11) and equation (12) in equation (1) and equation (11) in equation (2) yields the following system of differential equations that is solved numerically to obtain the optimal state variables:

$$\frac{d}{dt}I(t) = \hat{P}(t) + \frac{x_2(t)x_5(t) - 2x_1(t)x_4(t)}{4x_3(t)x_4(t) - x_5(t)^2} - D(t) - \theta(t)I(t), \tag{13}$$
$$\frac{d}{dt}Q(t) = -\left[\lambda(t) + \hat{u}(t) + \frac{x_1(t)x_5(t) - 2x_2(t)x_3(t)}{4x_3(t)x_4(t) - x_5(t)^2}\right]Q(t) + \left[\hat{P}(t) + \frac{x_2(t)x_5(t) - 2x_1(t)x_4(t)}{4x_3(t)x_4(t) - x_5(t)^2}\right]\alpha(t). \tag{14}$$

The optimal objective function value, where the " $(\boldsymbol{t})$ " has been omitted for conciseness, is given by

$$\begin{aligned} J^* = \quad & x_0 + \frac{1}{(x_5^2 - 4x_3)^2}\left[(2x_1x_4 - x_2x_5)\left(x_1x_5^2 - 2x_1x_3x_4 - x_3x_2x_5\right)\right. \\ & +(2x_3x_2 - x_1x_5)\left(x_2x_5^2 - 2x_3x_2x_4 - x_1x_4x_5\right) \\ & \left.+x_5(2x_1x_4 - x_2x_5)(2x_3x_2 - x_1x_5)\right]. \end{aligned} \tag{15}$$

4. Numerical Example

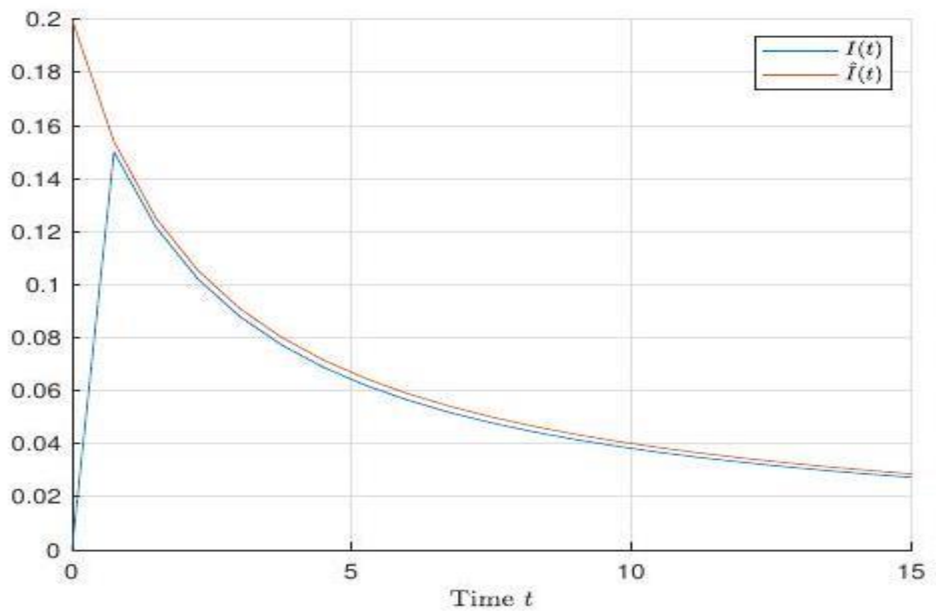
To illustrate the results obtained, consider a manufacturing firm on a lake as described in this paper. Initially, at time $t_0 = 0, I_0 = 0, Q_0 = 0$, and the firm is seeking the optimal production rate $P(t)$ and the optimal rate of expenditure for pollution control $u(t)$, over a prediction horizon of length $T = 15$. The firm has selected the following targets for the state variables: $\hat{I}(t) = 1/(5 + 2t)$ and $\hat{Q}(t) = 1/(25 + 50t^2)$. It follows from equation (3) and equation (4) that the optimal targets for the control variables are given by

$$\hat{P}(t) = \frac{-2}{(5 + 2t)^2} + \frac{1}{1 + t} + \frac{1}{(1 + 5t)(5 + 2t)}$$

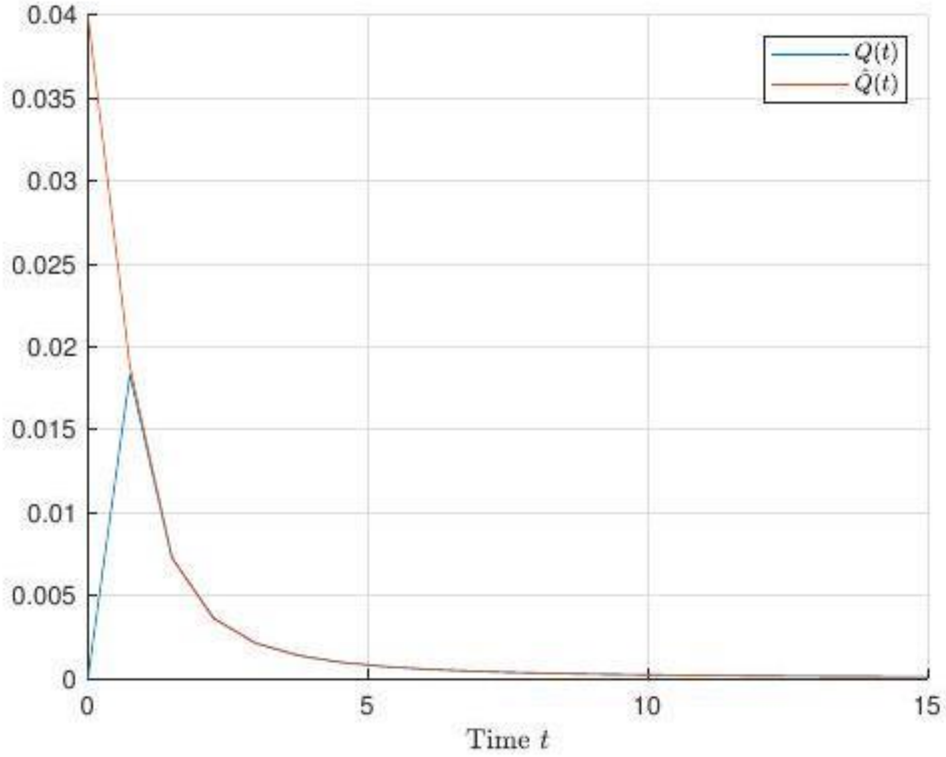
and

$$\hat{u}(t) = \frac{100t}{25 + 50t^2} - \frac{1}{(1 + 10t)} + \frac{25 + 50t^2}{1 + 15t} \times \left[\frac{-2}{(5 + 2t)^2} + \frac{1}{1 + t} + \frac{1}{(1 + 5t)(5 + 2t)}\right].$$

To minimize the gap between each variable and its corresponding target, the firm estimates that the penalties due to the deviations are given by $p_1 = 1, p_2 = 2,$ $q_1 = 1, q_2 = 2, c_1 = 40,$ and $c_2 = 45$. Estimating the exogenous variables, the firm finds that the self-cleaning ability rate of the water is given by $\lambda(t) = 1/(1 + 10t)$, and the pollutivity of a unit production is given by $\alpha(t) = 1/(1 + 15t)$. Items on the shelf deteriorate at a rate $\theta(t) = 1/(1 + 5t)$, and finally the demand rate is forecast to be $D(t) = 1/(1 + t)$. The prediction interval [0,15] is divided into $m = 20$ subintervals of equal length $h = 0.75$. Using MATLAB, the programming language and numeric computing environment developed by MATHWORKS, the firm solves numerically the nonlinear differential system equation (13) and equation (14). The solution obtained is depicted in Figure 1. We can see that each state variable converges towards its target, as desired. The optimal state variables thus obtained are then substituted in the control equations equation (11) and equation (12). The resulting optimal control variables are depicted in Figure 2. We can see that each control variable converges towards its target, as desired. The optimal objective function value is found to be $J^* = 17{,}962$. In summary, all the firm has to do is decide what their goal inventory level and goal lake quality should be. Then, they readily know, at each instant of time, how much they should produce, and how much they should spend to control the pollution.

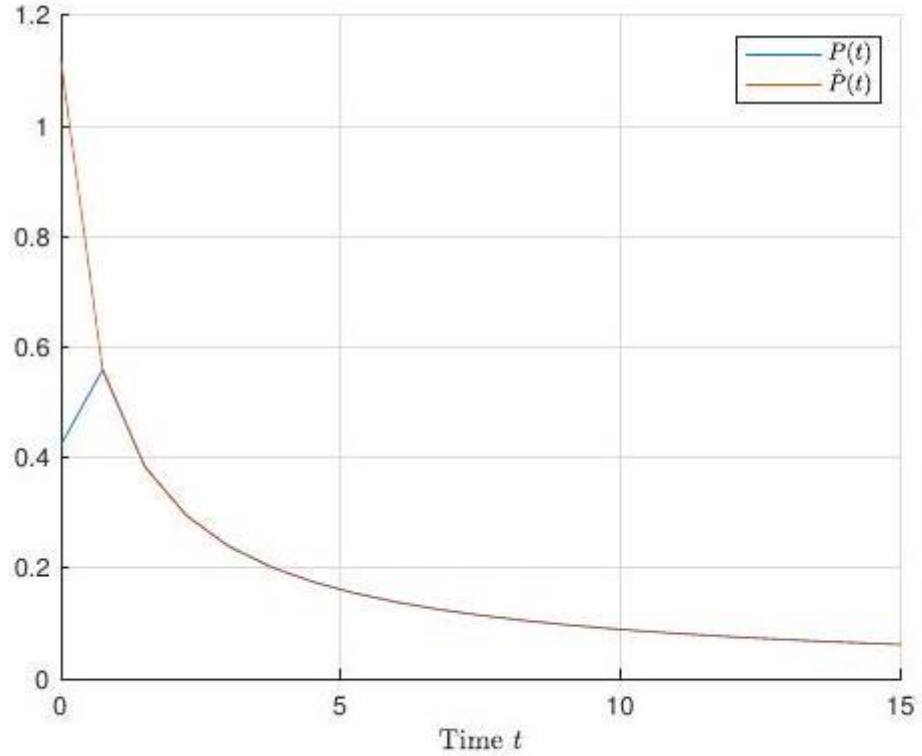


(a) Optimal state variable $I(t)$

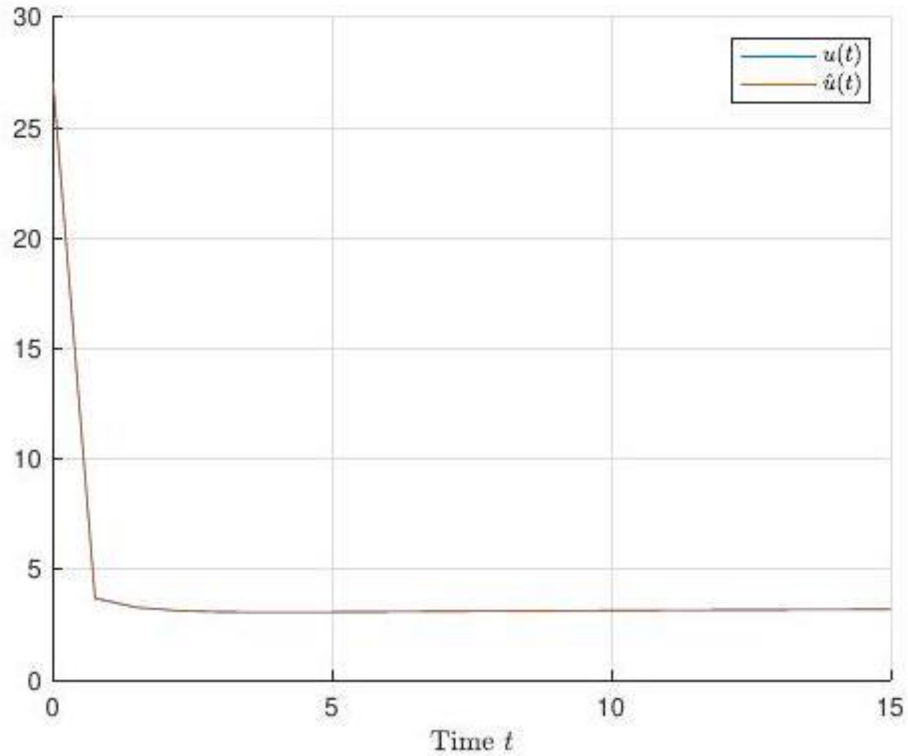


(b) Optimal state variable $Q(t)$

Figure 1: Optimal inventory level and quality of the lake.



(a) Optimal control variable $P(t)$



(b) Optimal control variable $u(t)$

Figure 2: Optimal production rate and rate of expenditure for pollution control.

The method works quite well for small values of the prediction interval. To confirm this, we have calculated the optimal objective function value J^* for different values of the prediction horizon length T. Figure 3 shows that, as expected, the higher the value of T, the higher the total cost.

5. Conclusion

The present paper contributes to the greening of a manufacturing firm. The problem considered uses a linear state equation to model the variations of the inventory level and a nonlinear equation to model the quality of the water. The optimal production rate and the optimal expenditure rate to control the water pollution are obtained.
As a future work, one could consider the case where the deterioration rate of the items depends not only on time but also on the quality of the lake. State dependent usually renders the model more complex and mathematically more challenging.

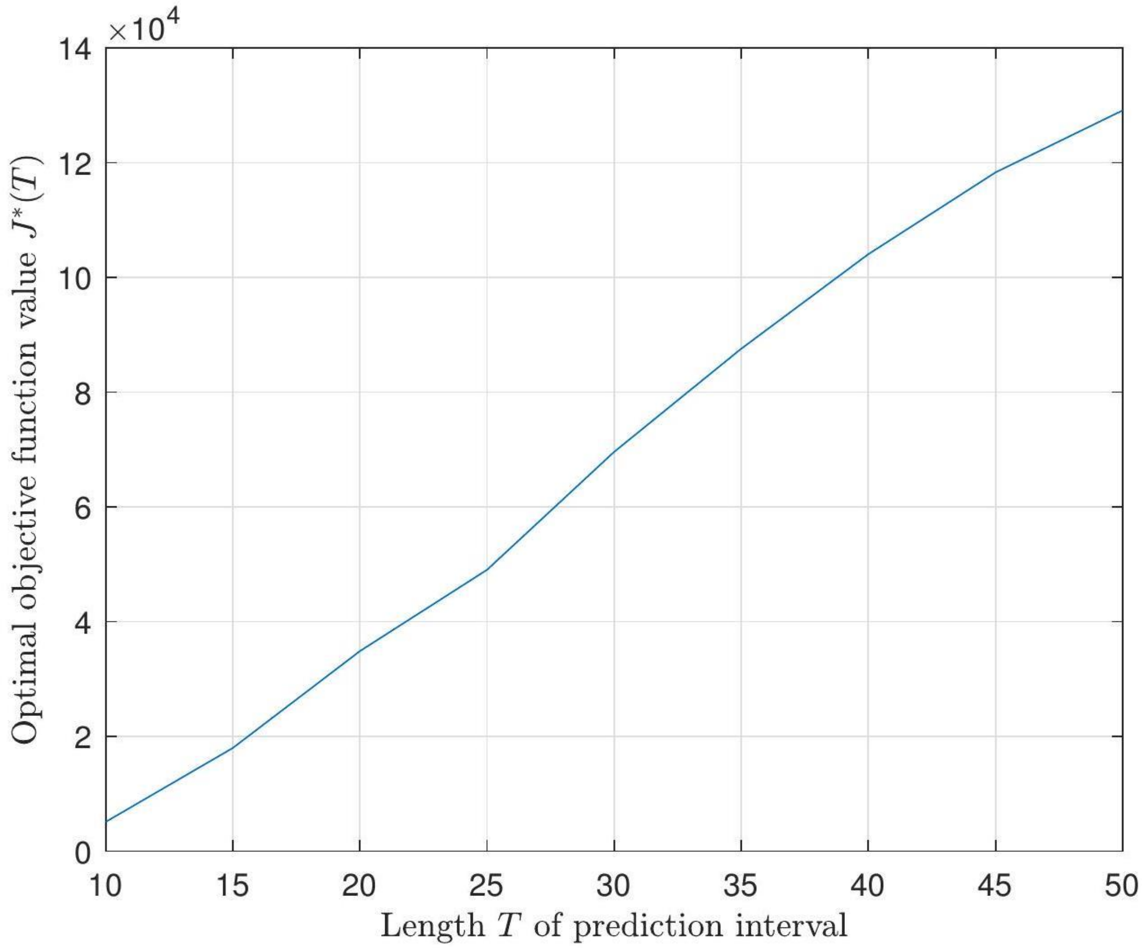


Figure 3: Optimal objective function value $J^*(T)$.

References

Brock, W. A., & Dechert, W. D. (2008). The polluted ecosystem game. *Indian Growth and Development Review*. https://doi.org/10.1108/17538250810868116.

Clarke, H. (1987). Production planning in a firm subject to environmental regulation. *Resources and Energy*, 9(1), 35–44. https://doi.org/10.1016/0165-0572(87)90021-1

Da Silva, A. A., & Souza, F. G. (2008). The economics of water resources for the generation of electricity and other uses. *Annals of Operations Research*. https://doi.org/10.1007/s10479-008-0360-z

Feichtinger, G. (1982). The Nash solution of a maintenance-production differential game. *European Journal of Operational Research*, 10(2), 165–172. https://doi.org/10.1016/0377-2217(82)90156-4

Feichtinger, G. (1988). Production-Pollution Cycles. *Lecture Notes in Economics and Mathematical Systems*, 263–276. https://doi.org/10.1007/978-3-642-46631-1_18

Feichtinger, G., & Jørgensen, S. (1983). Differential game models in management science. *European Journal of Operational Research*. https://doi.org/10.1016/0377-2217(83)90308-9

Hartl, R. F. (1982). Optimal control of concave economic models with two control instruments. *Springer eBooks*, 227–245. https://doi.org/10.1007/978-94-009-7901-7_18

Hartl, R. F. (1983). Optimal maintenance and production rates for a machine. *Journal of Economic Dynamics and Control*, 6, 281–306. https://doi.org/10.1016/0165-1889(83)90055-6

Hartl, R. F., Kirakossian, G., Hakobyan, S. R., & Markaryan, A. H. (2006). A dynamical model of water recycling in a mineprocessing enterprise. *Central European Journal of Operations Research*, 14(1), 45–57. https://doi.org/10.1007/s10100-006-0159-4

Highfill, J., & McAsey, M. (2001). An Application of Optimal Control to the Economics of Recycling. *Siam Review*, 43(4), 679–693. https://doi.org/10.1137/s0036144599362698

Jørgensen, S., Martín-Herrán, G., & Zaccour, G. (2010). Dynamic Games in the Economics and Management of Pollution. *Environmental Modeling & Assessment*, 15(6), 433–467. https://doi.org/10.1007/s10666-010-9221-7

Kawaguchi, K. (2003). Optimal Control of Pollution Accumulation with Long-Run Average Welfare. *Environmental and Resource Economics*, 26(3), 457–468. https://doi.org/10.1023/b:eare.0000003609.50884.41

Kossioris, G. T., Plexousakis, M., Xepapadeas, A., De Zeeuw, A., & Mäler, K. (2008). Feedback Nash equilibria for non-linear differential games in pollution control. *Journal of Economic Dynamics and Control*, 32(4), 1312–1331. https://doi.org/10.1016/j.jedc.2007.05.008

Nævdal, E. (2001). Optimal Regulation of Eutrophying Lakes, Fjords, and Rivers in the Presence of Threshold Effects. *American Journal of Agricultural Economics*, 83(4), 972–984. https://doi.org/10.1111/0002-9092.00223

Sethi, S. (1977). A linear big-bang model of firm behavior and water quality. *IEEE Transactions on Automatic Control*, 22(5), 706–714. https://doi.org/10.1109/tac.1977.1101596

Sethi, S. (2019). Optimal Control Theory. Springer eBooks. https://doi.org/10.1007/978-3-319-98237-3

Tapiero, C. S. (2005a). Environmental quality control and environmental games. *Environmental Modeling & Assessment*, 9(4), 201–206. https://doi.org/10.1007/s10666-005-2451-4

Tapiero, C. S. (2005b). Environmental quality control: a queueing game. *Stochastic Environmental Research and Risk Assessment*, 19(1), 59–70. https://doi.org/10.1007/s00477-004-0206-4

Tapiero, C. S. (2009). Energy consumption and environmental pollution: a stochastic model. *Ima Journal of Management Mathematics*. https://doi.org/10.1093/imaman/dpn041