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The Best Compromise Solution for Multi-objective Programming Problems

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Abstract

This paper focuses on multi-objective optimization problems that are an important part of operations research. This part is concerned with mathematical optimization problems involving two or more interdependent objectives to be optimized simultaneously. Thus, there is not a single optimal solution for multi-objective problems, but a set of solutions that represents the compromise (Pareto-optimal, efficient, non-dominated, trade-off, or non-inferior) solutions and can be visualized as Pareto front in the objective space. The best solution of this set has the shortest distance to the ideal (utopian) solution, whereas the ideal solution optimizes all objective functions, which often cannot be found. The main contribution of this paper is to introduce some methods to find the best compromise solution. These methods depend on new calculations for the normal of objectives. They can help to reduce the overall computational distance of the searching process. Therefore, they are flexible and stable. Besides, some numerical examples are presented to demonstrate the effectiveness of proposed methods with discussing their similarities and differences. The experimental results show that the proposed methods are effective and efficient for many different multi-objective (convex and non-convex) problems.

Keywords: Multi-objective optimization problems, Ideal point, Best compromise solution, The advanced Alia's method, The mixed Alia's method.

1. Introduction

In recent years, there has been a quickening pace of events and increased growth in the needs or demands of society with conflicting objectives or opinions to each other that led to the creation of many different applications of multi-objective optimization problems (MOPs). Consequently, the development of systems efficiency has become an attractive research topic. Most of these recent improvements are concentrating on traditional and evolutionary multi-objective optimization problems. A multi-objective optimization or vector optimization method is a method for optimizing the collection of objective functions subject to a number of constraints that are bounded systematically and simultaneously. There are two goals of such a problem: first, finding a set of solutions as close as possible to Pareto-optimal front (trade-off solutions), and second finding a set of solutions as diverse as possible in the obtained non-dominated front. Generally, a multiple-objective optimization problem doesn't have a single solution that could optimize all objectives simultaneously. It never searches for an optimal solution but for an efficient solution that can best suit a compromise solution to all multiple objectives. Getting a suitable compromise solution corresponding to a multi-objective optimization problem is a difficult task due to the conflict between various objectives and goals. However, there is a certain area where mathematical modeling and programming needed (Ojha, 2009), (Mavrotas, 2009), (Davoodi, 2011), (Ota, 2015). This research is considered an interesting extension of the advanced Alia's methods (Gebreel, 2021), where it proposes to solve the multi-objective optimization problems for providing the best solution that is very close to the utopian point. The paper is divided into four main parts; titled methods, theorems, features, and examples.

The remainder of this paper is organized as follows: the next section covers the basic concepts and definitions of multi-objective optimization. After that, section 3 discusses the proposed four models to calculate the normal of objectives. Followed by, section 4 introduces in more details the proposed methods to find the best compromise solution. Section 5 illustrates some numerical examples and presents a comparison with previous work. Finally, section 6 highlights the main conclusions and future work of the paper.

2. Multi-objective Optimization

A multi-objective optimization problem can be mathematically defined as follows:

Min: $F(x) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x})), \quad k \geq 2,$

Subject to: $M = \{\mathbf{x} \in R^n / g_r(\mathbf{x}) \leq 0, r = 1, 2, 3, \ldots, m\}.$ (1)

Where:

$f_i(\mathbf{x})$, and $g_r(\mathbf{x})$, $i = 1, 2, \ldots, k$, $r = 1, 2, \ldots, m$ are continuous functions,

$\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a decision vector that represents a solution in the search-space of n dimensions,

k is a number of objectives,

The set of constraints "M" of the problem defines the feasible region in the search space of the problem. Any vector of variables "**x**" which satisfies all constraints is considered a feasible solution. The goal is minimizing all objective functions simultaneously.

Assume that

$f_i(\mathbf{x}^*) = \min f_i(\mathbf{x}), \quad i = 1, 2, \ldots, k,$

Subject to: $\mathbf{x} \in M$. (2)

The following basic terms related to multi-objective optimization that will be frequently used in further discussion.

2.1. Scalarization

Scalarization is a standard technique to find Pareto optimal points for a vector optimization problem. It works on combining the objective functions to become a single objective function that will be optimized (Kashfi, 2010), (Boyd, 2004).

2.2. Convex and Non-convex Optimization Problem

A convex optimization problem consists of both objectives and constraints, which are mainly convex. This means that they satisfy the inequality:

$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y)$, for all $x, y \in R^n$, and all $\alpha, \beta \in R$ with $\alpha + \beta = 1, \alpha \geq 0, \beta \geq 0$. (3)

Therefore, linear and quadratic programming problems are both considered to be convex problems. However, any problem will be considered as non-convex, when it has a non-convex objective or a non-convex constraint as pictured below in **Figure 1**. Such a problem may have multiple feasible regions and multiple locally optimal points within each region (Gebreel, 2021), (Boyd, 2004).

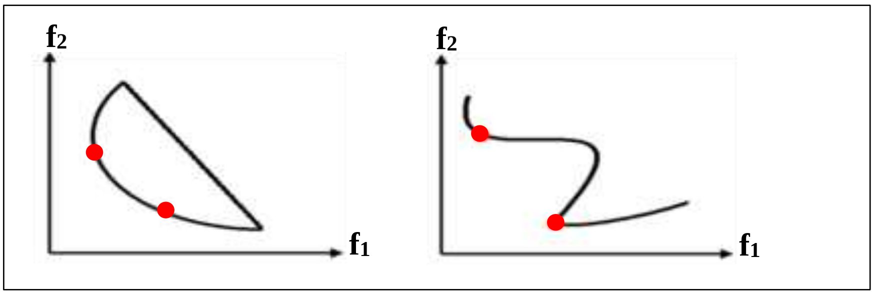

Figure 1: Convex and Non-convex Shapes.

2.3. Nonlinear Optimization

Nonlinear optimization (or nonlinear programming) is the term used to describe an optimization problem when the objective or constraint functions are not linear (Boyd, 2004). The nonlinear programming contains quadratic and non-convex optimization problems.

2.4. The standard Euclidean distance method

The standard Euclidean distance between two points in a bi-dimensional space is the square root of the sum of the squared differences between the first and second components of each point (Kamal, 2018), (Opricovic & Tzeng, 2004). Giving two points (x_1, x_2) and (y_1, y_2), their Euclidean distance is calculated as:

$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. (4)

Extension to the n -dimension space, the formula is as follows:

$$\sqrt{\sum_{j=1}^{n}\left(x_j - y_j\right)^2} \quad (5)$$

2.5. The best efficient solution

The best efficient point on the efficient front is called ***the Best point*** which has the shortest distance from the utopian point (Gebreel, 2021).

2.6. Pay-off table

A pay-off table (pay-off matrix) is constructed using the decision vectors obtained when calculating the utopian objective vector. Column i of the pay-off table displays the values of all objective functions calculated at the point, where f_i obtained its minimal value. Hence, f_i^* is at the main diagonal of the table. The maximal value of the row i in the pay-off table can be selected as an estimation of the upper bound of the objective f_i for i =1, 2, 3, …, k over the efficient set (Das, 1999), (Miettinen. 1998).

2.7. Alia's normal model (ANM)

It helps to find the best efficient solution for multi-objective convex programming problems.

2.8. The advanced Alia's method

On solving the multi-objective convex programming problems, this method gives the best compromise solution for all or some weights of objectives. It develops the existing Alia's method based on Alia's normal model (ANM).

2.9. The mixed Alia's method

It integrates the advanced Alia's method with the distance of objectives method (Gebreel, 2021). Its mathematical model is as follows:

(MAP):

Min: $\left(\sum_{i=1}^{k} w_i\, d\, |\, f_i(\mathbf{x}) - f_i^*| + \|N\|^2\, \delta\right)$,

Subject to:

$f_i(\mathbf{x}) - n_i\, \delta - \sum_{i=1}^{k} w_i\, d \le f_i^*$, i = 1, 2, 3, ..., k,

$M = \{\mathbf{x} \in R^n \,/\, g_r(\mathbf{x}) \le 0,\ \ r = 1, 2, \ldots, m\}$. (6)

Where:

$\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is an n-vector of decision variables,

$w_1, w_2, \ldots, w_k$ are the weights of the objective functions $f_i(\mathbf{x})$, $w_i \ge 0$, i = 1, 2, ..., k, $\sum_{i=1}^{k} w_i = 1$.

d is a general deviational variable for all objectives.

k is a number of objective functions.

f_i^*, i = 1, 2, 3, …, k are the individual optimal of the objectives.

$N\delta$ is the normalized controlling vector.

δ (variable) is clearly positive due to the feasibility of the constraints.

$N = (n_1, n_2, \ldots, n_k)$ is the normal vector directed in the positive direction to the utopia hyper-plane.

3. The Proposed Normal of Objectives

This section presents some new techniques for calculating the normal of objectives. There are four formulations as follows:

3.1. The First Alia's Normal Formulation

The objective function of the first Alia's normal formulation **(FAN)** consists of the standard Euclidean distance for the normal vector of objectives from the utopia point. Additionally, its constraint is the same constraint as Alia's normal formulation **(ANM)** (Gebreel, 2021) for k objectives. This formulation is given below:

(FAN):

Min: $\sqrt{\sum_{i=1}^{k}(n_i - f_i^*)^2}$,

Subject to: $p\,\mathbf{t} - \sum_{i=1}^{k} n_i \le f_i^*$, i = 1, 2, 3, ..., k. (7)

Where:
p is a pay-off matrix.
t is a vector of the decision variables for normal of objectives.

Theorem 1:
Let n_i, i = 1, 2, 3, …, k are the normal of objectives vector, f_i^* are the individual optimal of objectives f_i, p is a pay-off matrix, then the solution of ***(FAN)*** *problem must be optimal.*
Proof:
In the **(FAN)** problem, the standard Euclidean distance between the normal of objectives (n_i, i =1, 2, 3, …, k) and the individual optimal of objectives f_i are optimized, such that: $p\,\mathbf{t} - \sum_{i=1}^{k} n_i \leq f_i^*$. Then based on the optimality linear programming theory, the optimal solution of this problem is given.

3.2. The Second Alia's Normal Formulation

Basically, this model is used to distinguish between conflicting and non-conflicting objectives. It optimizes the pay-off matrix and normal of objectives that are limited by the optimum value for every objective function. The obtained results are zeros or the absolute value of the individual optimal solution of objectives. The non-conflicting objective has a zero value of n_i, whereas the conflicting objective has n_i equal to the absolute value of the optimum value for each objective. The second Alia's normal problem **(SAN)** can be represented as follows:
(SAN):

Min: C,
Subject to: $p\,\mathbf{t} - \sum_{i=1}^{k} n_i \leq f_i^*$, i = 1, 2, 3, ..., k. (8)

Where, C is any other variable for helping to optimize the constraint of problem **(SAN)**. Of course, its value always is equal to zero.

Lemma 1:

The smallest face containing all the edges incident to a common vertex of a pointed polyhedron is the all polyhedron (Armand, 1993).

Theorem 2:

The optimized normal of objectives with the pay-off matrix which are limited by the individual optimal solution of objectives are containing a polyhedron of problem ***(SAN)****.*

Proof:

Based on **Lemma 1**, there exists an optimal solution for the linear programming problem **(SAN)**. Then, the proof is obtained.

Remark 1:

It is clear that, this model presents many zeros for values of n_i, which creates flexibility in selecting the appropriate values of the normal.

3.3. The Third Alia's Normal Formulation

This model avoids the defects of the previous two models to improve their results. Therefore, its results have high quality to find the best compromise solution easily. It optimizes the total variables of this problem. The constraint of normal problem consists of a pay-off matrix and the normal of objectives; that are limited by the individual optimal solutions. The third Alia's normal **(TAN)** problem can be represented as follows:

(TAN):
Min: $(\mathbf{t} + \sum_{i=1}^{k} n_i)$,

Subject to: $p\,\mathbf{t} - \sum_{i=1}^{k} n_i \leq f_i^*$, i = 1, 2, 3, ..., k. (9)

Theorem 3:

On optimizing all variables of the normal problem with its constraint, the values of n_i are said to be of high quality to get the best solution easily. An illustration of such a constraint is as follows:

$p\,\mathbf{t} - \sum_{i=1}^{k} n_i \leq f_i^*$, i = 1, 2, 3, ..., k. (10)

Proof:

The theorem is proved by applying the linear programming theory of the **(TAN)** problem.

Remark 2:

For the above three normal formulations, the resulting normal in the minimum case has the minimum value corresponding to the individual optimal of objectives f_i^*.

3.4. Alia's normal formula (ANF)

In a convex multi-objective problem, this method uses the difference between the optimal of every objective function and the optimal of all objectives simultaneously as normal of objectives. It has the following form:

(ANF):

$$| \boldsymbol{f}^*(\boldsymbol{x_i}) - f_i^* |, \ i = 1, 2, 3, ..., k, \quad \text{(In the minimum case)} \tag{11}$$

$$| f_i^* - \boldsymbol{f}^*(\boldsymbol{x_i}) |, \ i = 1, 2, 3, ..., k, \quad \text{(In the maximum case)} \tag{12}$$

Where, $\boldsymbol{f}^*(\boldsymbol{x_i})$ is the optimal solution of the total objectives (without weights), and f_i^*, i = 1, 2, 3, ..., k are the individual optimal solutions.

Theorem 4:

In a convex multi-objective problem optimization, let the optimal solution of the total objectives (without weights) is $\boldsymbol{f}^*(\boldsymbol{x_i})$*, and the individual optimal of the objectives is* f_i^**, then the absolute difference between them (* $| \boldsymbol{f}^*(\boldsymbol{x_i}) - f_i^* |$ *) is used as normal of objectives to get the best solution of this problem.*

Proof:

Since the best compromise solution of a multi-objective problem has the shortest distance from the ideal point, then in a convex multi-objective problem, the difference between the individual optimal of the objectives and the optimal solution of them ($| \boldsymbol{f}^*(\boldsymbol{x_i}) - f_i^* |$) helps to get the best point. Where, this optimal point is one of the feasible known points toward the best solution to such a problem.

Remarks 3:

- When using Alia's normal model (ANM) in a multi-objective problem up to two objectives, the following can be observed:
  - This model is suitable for only two objective optimization problems. Where, it requires knowledge of the distinction between conflicting and non-conflicting objectives.
  - If all n_i, i = 1, 2, 3, …, k of objectives (conflicting and non-conflicting together) are optimized, the resulting values of zeros are increasing.
  - In conflicting and non-conflicting objectives optimization problem, the n_i of conflicting objective is selected with the non-conflicting objective that helps to get the best point easily.
- The first three normal formulations are different in objectives function.
- The maximum functions for the first three normal models formulations are the negative signal of minimum functions for them.

4. The Proposed Methods

This section designs some methods to solve the multi-objective problems for finding the best efficient solution. The main idea of these proposed methods comes from developing of the advanced Alia's method and the mixed Alia's method with some additional changes. In the proposed method, the second condition is considered as a robust condition for their design. It computes the *Euclidean distance* of objectives which minimizes the distances for all the obtained efficient points from the individual optima. In this case, the robust Pareto-optimal set among objectives can be given through searching for the best possible solution.

4.1. The steps of the proposed methodologies

The following steps can be used to implement the proposed methods:

1. Construct the Pay-off matrix (p) by calculating the individual minima (or maxima) f_i^* of the objectives f_i.
2. Determine ***Alia normal*** of objective vectors (N) by any one of the proposed normal models.
3. Formulate the multi-objective optimization model as single objective optimization with the use of weighting objectives method.
4. Solve the proposed formulations using any of the available solvers such as LINGO to obtain an efficient solution.

5- Calculate the Euclidean distance of this efficient point. If this solution has the minimum distance from the utopian point, then the process is terminated, otherwise proceed to next step.
6- Define new weights (or/ and sometimes new values of normal) for each objective and repeat from **step 3** to **step 6** until the best efficient solution is reached.

The flowchart of the proposed methods is given in the **Figure 2**.

In this work, two types of the proposed methods are introduced. The first type is related to develop the advanced Alia's method, but the second type develops the mixed Alia's method. Details about these types will be given below.

First of all, let us denote the Convex Hull of the Individual Minima in the objective space **(CHIM)** by $\boldsymbol{H}$, that is the image of constraints $\boldsymbol{C}$ mapped by the vector function $\boldsymbol{F}$ in the objective space. An element $\boldsymbol{P} \in \boldsymbol{H}$ is a vector (P_1, …, $P_{\mathbf{H}}$), where P_i is the i-th objective function value. The concept of best compromise solution is define as follows: Let **P** be a Pareto optimal solution for **(MOP)**; **P** is said to be a best compromise solution if P_{i} reaches a minimum distance value for i= 1, …, $\boldsymbol{H}$.

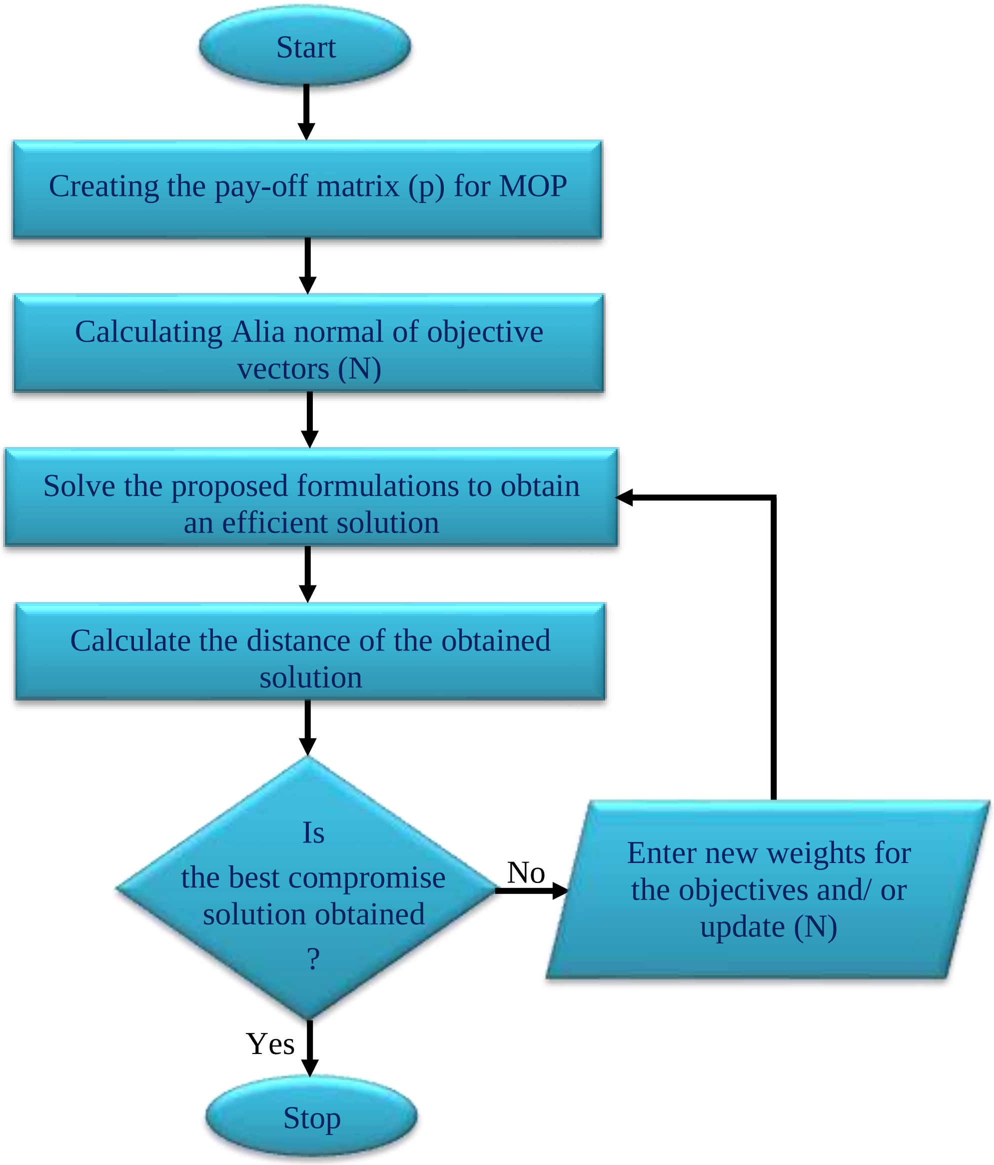

Figure 2: Flowchart Representation of the Proposed Methods.

4.2. The types of the proposed methodologies

In the following sections, three different formulations are presented to find the best efficient solution of a multi-objective problem for both convex and non-convex cases.

4.2.1. The first formulation (For convex case)

This formulation is a modified Alia method for studying the general convex multiple objective programming problem to get the best point. Consider the following convex problem denoted as **(AP1)**.

(AP1): Min $\left(\sum_{i=1}^{k} w_i f_i(\mathbf{x}) + \|N\|^2 \delta + h\right)$,

Subject to:

$f_i(\mathbf{x}) - n_i\, \delta \leq f_i^*$, i = 1, 2, 3, ..., k,

$\left| \sqrt{\sum_{i=1}^{k}(f_i - f_i^*)^2} - a\, h \right| \geq 0$, i = 1, 2, 3, ..., k,

$M = \{\mathbf{x} \in R^n / g_r(\mathbf{x}) \leq 0,\ r = 1, 2, 3, \ldots, m\}$. (13)

Where:

$\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a vector of the decision variables, n is a number of the decision variables.

$w_1, w_2, \ldots, w_k$ are the weights of the objective functions $f_i = f_i(\mathbf{x})$, $w_i > 0$, i= 1, 2, ..., k, $\sum_{i=1}^{k} w_i = 1$.

d is a general deviational variable for all objectives.

k is a number of objective functions.

f_i^*, i = 1, 2, 3, …, k are the optimum value of $f_i(\mathbf{x})$ over M.

$N\delta$ is the normalized controlling vector.

δ is a real variable; it is clearly positive due to the feasibility of the constraints.

$N = (n_1, n_2, \ldots, n_k)$ is the normal vector directed in the positive direction to the utopia hyper-plane.

a and h are the deviational variables for the distance = $\sqrt{\sum_{i=1}^{k}(f_i - f_i^*)^2}$ in the convex case.

Let us look at the following lemma and theorems, which are important for the interpretation of **(AP1)**.

Lemma 2:

Let the constraints of Alia's problem (AP) satisfy Slater constraint qualification (or any other constraint qualifications). If for $\overline{w} > 0$, $(\overline{x}, \overline{\delta})$ is an optimal solution of (AP), then $\overline{x}$ is an efficient solution of a multi-objective convex programming problem (MOCPP) (Gebreel, 2016).

Theorem 5:

If for $\overline{w} > 0$, $(\overline{x}, \overline{\delta})$ is an optimal solution of (AP) such that $f_i(\overline{x}) - n_i \overline{\delta} = f_i^*$, i = 1, 2, 3, ..., k, then $\overline{x}$ will be an Alia efficient point for (MOCPP) (Gebreel, 2016).

Theorem 6:

*A solution $\mathbf{x}^* \in M$ is an efficient solution of (MOCPP) if and only if $\mathbf{x}^*$ is an optimal solution of **(AP1)**.*

Proof:

Let for $\overline{w} > 0$, $(\overline{x}, \overline{n_i}, \overline{\delta})$ be an optimal solution of (**AP1**). Then from **Lemma 2**, it follows that $\overline{x}$ is an efficient solution of (MOCPP). Since $\overline{x}$ satisfies the relation $f_i(\mathbf{x}) - n_i\, \delta \leq f_i^*$, i = 1, 2, 3, ..., k, with selecting "$\overline{h}$" at optimum value such that:

$$\left| \sqrt{\sum_{i=1}^{k}(f_i - f_i^*)^2} - \overline{a}\, \overline{h} \right| \geq 0, \quad (14)$$

Then from the definition of ***efficient solution*** (Miettinen, 1998), it is clear that $\overline{x}$ is an ***efficient solution*** for (MOCPP).

Theorem 7:

*In the objective space $\boldsymbol{E^P}$ $(2 \leq \boldsymbol{P} \leq \infty)$, let all objectives in the case of (MOCPP) are conflicting to each other, an efficient solution $\mathbf{x}^*$ of problem **(AP1)** is said to be only one best compromise solution in $\boldsymbol{H} \subseteq \boldsymbol{F(C)}$ for any w > 0, if and only if there doesn't exist another solution $\boldsymbol{y} \in \boldsymbol{F(C)}$ such that $\boldsymbol{d(y)} > \boldsymbol{d(x^*)}$, and $\left| \sqrt{\sum_{i=1}^{k}(f_i - f_i^*)^2} - \bar{a}\, \bar{h} \right| \geq 0$. This mean $\mathbf{x}^*$ has minimum distance from the utopian point.*

Proof:

Firstly, the following Kuhn- Tucker (K.T) conditions (Bazzraa, 1979) to give the optimal solution are shown for this problem.

Kuhn- Tucker conditions for problem (AP1):

$$\sum_{i=1}^{k} w_i \frac{\partial f_i(\mathbf{x})}{\partial x_j} + \sum_{i=1}^{k} \mu_i \frac{\partial f_i(\mathbf{x})}{\partial x_i} + \sum_{i=1}^{k} \eta_i \frac{\sqrt{\sum_{i=1}^{k}(f_i - f_i^*)^2}}{\partial x_j} + \sum_{r=1}^{m} \alpha_r \frac{\partial g_r(\mathbf{x})}{\partial x_j} = 0,$$

$j = 1, 2, 3, \ldots, n,$ (15)

$\sum_{i=1}^{k} n_i \ \mu_i = \|N\|^2,$ $i = 1, 2, 3, ..., k,$ (16)

$\eta_i \, a = 1,$ $i = 1, 2, 3, ..., k,$ (17)

$f_i(\mathbf{x}) - n_i \delta \leq f_i^*,$ $i = 1, 2, 3, ..., k,$ (18)

$g_r(\mathbf{x}) \leq 0,$ $r = 1, 2, 3, \ldots, m,$ (19)

$\left| \sqrt{\sum_{i=1}^{k}(f_i - f_i^*)^2} - a\,h \right| \geq 0,$ $i = 1, 2, 3, ..., k,$ (20)

$\mu_i (f_i(\mathbf{x}) - n_i \delta - f_i^*) = 0,$ $i = 1, 2, 3, ..., k,$ (21)

$\alpha_r \, g_r(\mathbf{x}) = 0,$ $r = 1, 2, 3, \ldots, m,$ (22)

$\eta_i \left(\left| \sqrt{\sum_{i=1}^{k}(f_i - f_i^*)^2} - a\,h \right| \right) = 0,$ $i = 1, 2, 3, ..., k,$ (23)

$\mu_i \geq 0,$ $i = 1, 2, 3, ..., k,$ (24)

$\alpha_r \geq 0,$ $r = 1, 2, 3, ..., m,$ (25)

$\eta_i \geq 0,$ $i = 1, 2, 3, ..., k,$ (26)

Secondly, utilizing the result of **theorem 6** and assume that for another $w^* > 0$, (x^*, δ^*, h^*) is an optimal solution of problem (**AP1**) such that:

$f_i(x^*) - n_i \delta^* \leq f_i^*$, $i = 1, 2, 3, ..., k$, and (27)

$\left| \sqrt{\sum_{i=1}^{k}(f_i - f_i^*)^2} - a\,h \right| \geq 0,$ $i = 1, 2, 3, ..., k,$ (28)

Then, x^* is the best efficient solution of (MOCPP), and there is only one optimal solution of problem (**AP1**). Therefore, the minimum distance from the utopia point holds. The proof is completed.

Remarks 4:

1- *Uniqueness of the best point in convex optimization problems:* The best point in convex optimization problems is only one efficient solution regardless of the number of Pareto- optimal points in conflicting objectives.

2- Although sometimes the results of three Alia's normal formulations are different, but they can give the same best compromise solution.

3- *When the pay-off matrix of conflicting objectives is symmetric and the result of Alia normal model is zero(s)*, you can replace this zero(s) by one **(**or the total sum is one) to get the best point for all weights of problem (**AP1**). But, *if the pay-off matrix of conflicting objectives is not symmetric*, then the best point is resulted for some weights of objectives. Also, it can select any values of n_i, $i = 1, 2, \ldots, k$ for a quadratic problem that has conflicting and non-conflicting multi-objective more than two objectives.

Next, there are two models to find the best point in MOPs for some weights.

4.2.2. The second formulation (For convex case)

This formulation develops the mixed Alia's method. Consider the second following convex problem denoted as (AP2):

(AP2): Min $\left(\sum_{i=1}^{k} w_i \, d \,|\, f_i(\mathbf{x}) - f_i^* \,| + \|N\|^2 \delta + h\right)$,

Subject to:

$f_i(\mathbf{x}) - n_i \delta - \sum_{i=1}^{k} w_i \, d \le f_i^*$, $\quad i = 1, 2, 3, ..., k,$

$\left| \sqrt{\sum_{i=1}^{k} (f_i - f_i^*)^2} - a\,h \right| \ge 0$, $\quad i = 1, 2, 3, ..., k,$

$w_i \ge 0$, $i = 1, 2, 3, ..., k$, $\sum_{i=1}^{k} w_i = 1$.

$M = \{\mathbf{x} \in R^n / g_r(\mathbf{x}) \le 0, r = 1, 2, 3, \ldots, m\}$. (29)

Theorem 8:

If for some $\overline{w} \ge 0$, $(\overline{x}, \overline{\delta}, \overline{h})$ *is an optimal solution of problem* ***(AP2)*** *such that* $\left| \sqrt{\sum_{i=1}^{k} (f_i - f_i^*)^2} - a\,h \right| \ge 0$, *i = 1, 2, 3, ..., k, a and h are the deviational variables for the distance* $\sqrt{\sum_{i=1}^{k} (f_i - f_i^*)^2}$, *then* $\overline{x}$ *will be a* ***best efficient point*** *for a multi-objective convex programming problem.*

Proof:

Formulating the Kuhn- Tucker (K.T) conditions to get the optimal solution for this problem as follows:

Kuhn- Tucker conditions for problem (AP2)

$$\sum_{i=1}^{k} w_i \, d \, \frac{\partial \,|\, f_i(\mathbf{x}) - f_i^* \,|}{\partial x_j} + \sum_{i=1}^{k} \mu_i \frac{\partial f_i(\mathbf{x})}{\partial x_j} + \sum_{i=1}^{k} \eta_i \frac{\partial \sqrt{\sum_{i=1}^{k} (f_i - f_i^*)^2}}{\partial x_j} + \sum_{r=1}^{m} \alpha_r \frac{\partial g_r(\mathbf{x})}{\partial x_j} = 0,$$

$j = 1, 2, 3, \ldots, n,$ (30)

$\sum_{i=1}^{k} w_i \,|\, f_i(\mathbf{x}) - f_i^* \,| = \mu_i \sum_{i=1}^{k} w_i$, (31)

$\sum_{i=1}^{k} n_i \; \mu_i = \|N\|^2$, (32)

$\eta_i \, a = 1$, $\quad i = 1, 2, 3, ..., k,$ (33)

$f_i(\mathbf{x}) - n_i \delta - \sum_{i=1}^{k} w_i \, d \le f_i^*$, $\quad i = 1, 2, 3, ..., k,$ (34)

$g_r(\mathbf{x}) \le 0$, $\quad r = 1, 2, 3, \ldots, m,$ (35)

$\left| \sqrt{\sum_{i=1}^{k} (f_i - f_i^*)^2} - a\,h \right| \ge 0$, $\quad i = 1, 2, 3, ..., k,$ (36)

$\mu_i \left(f_i(\mathbf{x}) - n_i \delta - \sum_{i=1}^{k} w_i \, d - f_i^*\right) = 0$, $\quad i = 1, 2, 3, ..., k,$ (37)

$\alpha_r \, g_r(\mathbf{x}) = 0$, $\quad r = 1, 2, 3, \ldots, m,$ (38)

$\eta_i \left(\left| \sqrt{\sum_{i=1}^{k} (f_i - f_i^*)^2} - a\,h \right| \right) = 0$, $\quad i = 1, 2, 3, ..., k,$ (39)

$\mu_i \ge 0$, $\quad i = 1, 2, 3, ..., k,$ (40)

$\alpha_r \ge 0$, $\quad r = 1, 2, 3, ..., m,$ (41)

$\eta_i \ge 0$, $\quad i = 1, 2, 3, ..., k,$ (42)

Let for some $\overline{w} \geq 0$, ($\overline{\mathrm{d}}$, $\overline{x}$, $\overline{\delta}$, $\overline{\mathrm{h}}$, $\overline{\mathrm{a}}$) be an optimal solution of problem (**AP2**), and $\bar{n}_i$ is an optimal solution of the first, second, or third Alia normal model. Then from Lemma 2, it follows that $\overline{x}$ is an efficient solution of problem (**AP2**).

Let $\overline{x}$ satisfies the relations $f_i(\mathbf{x}) - n_i\,\delta - \sum_{i=1}^{k} w_i\, d \leq f_i^*$, and $\left|\sqrt{\sum_{i=1}^{k}(f_i - f_i^*)^2} - a\,h\right| \geq 0$, i = 1, 2, 3, ..., k. Since the problem **(AP1)** is part of the problem **(AP2)**. Then from the definition of ***the best efficient solution***, it is clear that $\overline{x}$ is the best efficient solution for some $\overline{w} \geq 0$.

Remarks 5:

1- When the selecting of values for the normal is correct in the model **(AP2)**, the best compromise solution is obtained for all weights.
2- If pay-off matrix of non-conflicting objectives is not symmetric and the result of Alia normal model(s) is zero(s), then the best solution from model **(AP2)** is obtained for all weights ($w_i \geq 0$, i = 1, 2, 3, …, k).
3- When the resulted value of n_i, i = 1, 2, 3, …, k from Alia's normal formulations is zero and the pay-off matrix is symmetric, then the best solution is obtained for all or some weights of objectives.
4- The first normal formulation **(FAN)** is used on the mixed method only, because it is not useful in Alia's method or the developed Alia's method.

Corollary:

Given the second method **(AP2)**, the weights in its constraints to get the best efficient solution can also be used as the normal for the first Alia's method **(AP1)**.

Proof:

Since the first Alia's method **(AP1)** works with the convex multi-objective problems, then it can use its normal (n_i, i = 1, 2, 3, …, k) with total values, which are equal to one (i.e.; $\sum_{i=1}^{k} n_i = 1$). Besides, the total weights of constraints in the second method **(AP2)** must be equal to one for the same problem. Thus, the proof is given.

4.2.3. The third formulation (For non-convex case)

This formulation solves the multi-objective non-convex programming problems, and it is denoted as **(AP3)**. It is considered as a part of **(AP2)** formulation. Thus, the second formulation **(AP2)** can be reduced to:

(AP3): Min $\left(\sum_{i=1}^{k} w_i \,|\, f_i(\mathbf{x}) - f_i^* \,| + \|N\|^2\,\delta + h\right)$,

Subject to:

$$f_i(\mathbf{x}) - n_i\,\delta \leq f_i^*, \qquad i = 1, 2, 3, ..., k,$$

$$\left|\sqrt{\sum_{i=1}^{k}(f_i - f_i^*)^2} - a\,h\right| \geq 0,$$

$$\mathbf{x} \in M. \tag{43}$$

Where:

$w_1, w_2, \dots, w_k$ are the weights of the objective functions $f_i(x)$, $w_i \geq 0$, i = 1, 2, ..., k, $\sum_{i=1}^{k} w_i = 1$.
$\|N\|^2 = n_1^2 + n_2^2 + \dots + n_k^2$.

Note that:

- $\|N\|^2 = n_1^2 + n_2^2 + \dots + n_k^2$. This formula is used for both cases (convex and non-convex problems).
- $\|N\| = n_1 + n_2 + ... + n_k$. This formula is used only in the convex case; because it may give unbounded solution in non-convex case.
- **In convex case**: a and h are the deviational variables for the distance = $\sqrt{\sum_{i=1}^{k}(f_i - f_i^*)^2}$, i = 1, 2, 3, ..., k.
- **In non-convex case:** a is scalar, and h is the deviational variable for the distance = $\sqrt{\sum_{i=1}^{k}(f_i - f_i^*)^2}$, i = 1, 2, 3, ..., k.

Theorem 9:

*If point x^* is an optimal solution for problem **(AP3)**, then x^* is best efficient solution in the case of non-convex problem (MOP) for some weights ($w_i \geq 0$, i = 1, 2, 3, ..., k, $\sum_{i=1}^{k} w_i = 1$).*

Proof:

Assume that the best efficient solution of the non-convex problem (MOP) for some weights ($w_i \geq 0$, i = 1, 2, 3, ..., k, $\sum_{i=1}^{k} w_i = 1$), which is gained by **(AP3)** model, is denoted x^*. Obviously, the result will be as follows: $f_i(x) > f_i(x^*)$, i =1, 2, 3, …, k, in the convex search space, $\forall\, x \in X$. Next, supposing that x^* is not the best efficient solution; in this case, there will be another solution $\bar{x} \in X$ such that $f_i(\bar{x}) < f_i(x^*)$, i = 1, 2, 3, …, k, $\forall\, \bar{x} \in X$. According to the assumption that the weighting coefficients w_i are nonnegative, the following statement will be:

$$\left(\sum_{i=1}^{k} w_i \,|\, f_i(\bar{x}) - f_i^* \,| + \|N\|^2 \delta + h \right) < \left(\sum_{i=1}^{k} w_i \,|\, f_i(x^*) - f_i^* \,| + \|N\|^2 \delta + h \right).$$

This is a contradiction to the assumption that x^* is a solution of this problem. Since the problem (MOP) at hand is non convex, then x^* is the best efficient solution for some weights of this problem.

Note that:

1- The first Alia normal **(FAN)** is not appropriate for the third formulation **(AP3)**.
2- The **(AP3)** formulation is more widely used for the non-convex problems than convex.
3- Alia's normal formula **(ANF)** is useful for the convex case rather than the non-convex case of multi-objective problems. Additionally, it is valid for the mixed advanced Alia's method rather than the advanced Alia's method.

Remarks 6:

1- The best solution of a non-convex case cannot be reached with any value of weights by weighting method.
2- The best point in the efficient set of any vector optimization problem (VOP) is given by the proposed methods, whether these objectives are conflicting or not conflicting.
3- The first formulation **(AP1)** produces the best solution in the convex case for all weights, when all the objective functions of problem (MOP) are conflicting with each other. However, if there are also non-conflicting objectives, then the best solution will be only for some weights. The other proposed formulations **(AP2**, and **AP3)** produces the best solution using only finite number of weights or all weights.
4- In the formulation **(AP2)**, the weights of objective functions may be different from the weights of constraints. On the other hand, the total of weights for objectives or constraints must be equal to one.
5- The presented theorems reveal that when the best point is given for the problem (MOP), it's basically considered an optimal solution of any one of the proposed methods.
6- The model **(AP2)** is more valid for **convex case** than non-convex case. However, it is possible to use "a" as scalar in the second constraint for the **non-convex problem**.
7- The maximum functions for the proposed methods are the negative signal of minimum functions for them.

4.2.4. The special formulations:

In this section, the family of special formulations from the proposed methods is introduced to find the best compromise solution for the multi-objective optimization problems. These special formulations are accurate alternative models.

(AP1-a):

Min: $\left(\sum_{i=1}^{k} w_i f_i(\mathbf{x}) + \|N\|^2 \delta \right)$,

Subject to:

$f_i(\mathbf{x}) - n_i \delta \leq f_i^*$, i = 1, 2, 3, ..., k,

$\sqrt{\sum_{i=1}^{k} (f_i - f_i^*)^2} \leq \mathbf{D}$, i = 1, 2, 3, ..., k,

$w_i \geq 0$, i = 1, 2, 3, ..., k, $\sum_{i=1}^{k} w_i = 1$.

$\mathbf{x} \in M$. (44)

(AP2-a):

Min: $\left(\sum_{i=1}^{k} w_i\ d\ |\ f_i(\mathbf{x}) - f_i^*\ | + \|N\|^2\ \delta\right)$,

Subject to:

$f_i(\mathbf{x}) - n_i\ \delta - \sum_{i=1}^{k} w_i\ d \le f_i^*$, $\quad i = 1, 2, 3, ..., k$,

$\sqrt{\sum_{i=1}^{k}(f_i - f_i^*)^2} \le \mathbf{D}$, $\quad i = 1, 2, 3, ..., k$,

$w_i \ge 0$, $i = 1, 2, 3, ..., k$, $\sum_{i=1}^{k} w_i = 1$.

$\mathbf{x} \in M$. (45)

(AP3-a):

Min: $\left(\sum_{i=1}^{k} w_i\ |\ f_i(\mathbf{x}) - f_i^*\ | + \|N\|^2\ \delta\right)$,

Subject to:

$f_i(\mathbf{x}) - n_i\ \delta \le f_i^*$, $\quad i = 1, 2, 3, ..., k$,

$\sqrt{\sum_{i=1}^{k}(f_i - f_i^*)^2} \le \mathbf{D}$, $\quad i = 1, 2, 3, ..., k$,

$w_i \ge 0$, $i = 1, 2, 3, ..., k$, $\sum_{i=1}^{k} w_i = 1$.

$\mathbf{x} \in M$. (46)

Where:

D (Constant) is the Euclidean distance between the optimal solution of total objectives (without weights) and the ideal point.

Note that:
In some nonlinear problems, the distance constraint in model **(AP3-a)** become:

$$\sqrt{\sum_{i=1}^{k}(f_i - f_i^*)^2} \ge 0, \quad i = 1, 2, 3, ..., k. \quad (47)$$

Remarks 7:

To obtain the best solution, the normal vector (N), the values of weight for objectives, a, and **D** have to be chosen carefully according to the problem.

4.3. The major features of the proposed methods

The proposed methods have some features are stated as following:

1) The solution obtained is efficient; that reflects the preferences of decision-maker.

2) An important task of these new methods is to find the best efficient solution. But also throughout this search, a set of efficent solutions closest to the utopian point is obtained. Thus, the decision-maker would be able to make a better and more reliable decision.

3) The weighted-sum of objectives is used here as a criterion for generating the efficient solutions until obtaining the best of them, where the weight vector provides information about what point on the Pareto-optimal front to converge.

4) Alia normal and Euclidean distance of objectives play also the most important role in determining the effectiveness and efficiency of the proposed methods in obtaining the best efficient solution.

5) The proposed methods differ from each other in their structure, but they find the robust Pareto-optimal solutions.

6) Due to the computational complexity of the nonlinear multi-objective problem, these models can give the best compromise solution after wards. Besides, the time of solution is increased when the number of objectives is increased rather than constraints.

7) Alia's normal formula (ANF) is used easily in the second method **(AP2)** for convex case of multi-objective problems.

5. Numerical Examples

For an illustration of the proposed methods, some examples are considered which include both convex and non-convex shapes. The formulated problems have been solved by Lingo software to obtain the best solution.

Example: 1

This linear example is introduced by Alia (Gebreel, 2016). It consists of two objectives and two variables of minimization problem as follows:

Min: F = ($f_1 = x$, $f_2 = y$**),**

Subject to:

$2x + y \geq 4$,

$2x + 3y \geq 8$,

$x + y \leq 4$,

$x \geq 0, y \geq 0$.

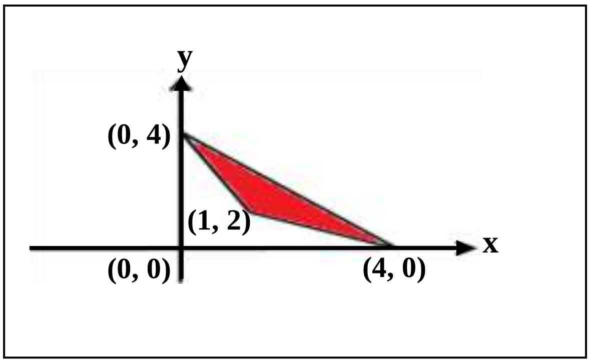

Figure 3: The decision and objective spaces of the first example.

The problem has two conflicting objectives which create a convex Pareto-optimal front as shown in **Figure 3**.

The ideal point of this example is (0, 0) that is infeasible solution. The pay-off matrix is $\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$, the first three Alia normal's models are $(0, 0)^T$, then Alia normal's values take zero or any positive values that achieve the best solution. However, if Alia normal's formula = $(f^*(x_i) - f_i^*)$ is equal to (3, 3), x_i= x, y, and i= 1, 2. Then the first formuation **(AP1)** is as follows:

Min: F= ($w_1 x + w_2 y + (n_1^2 + n_2^2)\,\delta + h$**),**

Subject to:

$x - n_1 \delta \leq 0$,

$y - n_2 \delta \leq 0$,

$2x + y \geq 4$,

$2x + 3y \geq 8$,

$x + y \leq 4$,

$\left| \sqrt{(x-0)^2 + (y-0)^2} - a\,h \right| \geq 0$,

$x \geq 0, y \geq 0$.

Since the normal N is $(n_1= 0, n_2= 0)^T$, the selected values of N that present the best point for all weights are $(n_1= 1.230769, n_2= 1.846154)^T$, and $(n_1= 0.1, n_2= 0.15)^T$ for some weights.

But, the second formuation **(AP2)** is:

Min: F= ($w_1\, d\, x + w_2\, d\, y + (n_1^2 + n_2^2)\,\delta + h$**),**

Subject to:

$x - n_1 \delta - w_1 d \leq 0$,

$y - n_2 \delta - w_2 d \leq 0$,

$2x + y \geq 4$,

$2x + 3y \geq 8,$

$x + y \leq 4,$

$\left| \sqrt{(x-0)^2 + (y-0)^2} - a\,h \right| \geq 0,$

$w_1 + w_2 = 1,\ x \geq 0,\ y \geq 0.$

The second formulation obtains the best point for all weights (w_1, w_2) > 0 when $(n_1= 0.1, n_2= 0.15)^T$ or $(n_1= 0.4, n_2= 0.6)^T$. The weights of objectives can be different from the value of weights in the constraints. But, the total values for every group of weights must be equal to one. When $n_1= n_2= 0$, this solution is obtained for some weights. In the special formulation, the value of D is 2.236068. Moreover, there are other two special formulations to get the best solution as follows:

Min: F= ($0.45\,x + 0.55\,y + 0.0325\,\delta$**),**

Subject to:

$x - 0.1\,\delta \leq 0,$

$y - 0.15\,\delta \leq 0,$

$2x + y \geq 4,$

$2x + 3y \geq 8,$

$x + y \leq 4,$

$\sqrt{(x-0)^2 + (y-0)^2} \leq 2.236068,$

$w_1 + w_2 = 1,\ x \geq 0,\ y \geq 0.$

Or

Min: F= ($0.463415\,d\,x + 0.536585\,d\,y + 5\,\delta$**),**

Subject to:

$x - \delta - 0.463415\ d \leq 0,$

$y - 2\,\delta - 0.536585\ d \leq 0,$

$2x + y \geq 4,$

$2x + 3y \geq 8,$

$x + y \leq 4,$

$\sqrt{(x-0)^2 + (y-0)^2} \leq 2.236068,$

$w_1 + w_2 = 1,\ x \geq 0,\ y \geq 0.$

The best point for all above models is: f_1^*= x= 1.230769, f_2^*= y= 1.846154, its Euclidean distance= 2.2188, and the total of objectives ($f_1^* + f_2^*$) is 3.076923. But, *Alia point* is (1.6, 1.6), and its distance= 2.263.

Example: 2

The considered multi-objective programming problem has the following form (taken from (Gebreel, 2016)):

Min: F = ($f_1 = x,\ f_2 = y,\ f_3 = -x - 3y,\ f_4 = 2x^2 - 4y$**),**

Subject to: $x + y \geq 2,$

$-x + y \leq 2,$

$3x + y \leq 6,$

$x \geq 0,\ y \geq 0.$

For this example, $f_1 = x$, $f_2 = y$, $f_3 = -x - 3y$, $f_4 = 2x^2 - 4y$, then
$f_1^* = 0$ attained at the point (0, 2),
$f_2^* = 0$ attained at the point (2, 0),
$f_3^* = -10$ attained at the point (1, 3),
$f_4^* = -10$ attained at the point (1, 3),

It is clear that f_3, f_4 are nonconflicting, and then we can omitt one of them from the problem. The calculated Alia normal is (1, 3, 0).

If f_4 is omitted, the best efficient solution is (0.9090909, 2.727273) based on the selected values of vector N = (1, 3, 1). It is given for all weights of model **(AP1)** and model **(AP3)**, $w_i \geq 0$, i = 1, 2, 3. But, this solution from **(AP2)** is obtained for all weights of objectives, $w_i > 0$, i = 1, 2, 3; when the values for weights of constraints are zero, zero, and one, respectively. Based on the first and third proposed normal models = (1, 3, 0), the model **(AP2)** uses weights of objectives = (0.818181818, 0.0, 0.181818182) and weights of constraints = (0.0, 0.0, 1.0) to give this best solution. The distance of best solution from the utopian point is 3.015. Also, this result is obtained when **D** = 3.16228 in the special models.

However, if f_3 is omitted, the best efficient solution is (0.3106, 2.3106). The distance of solution from the utopian point is 2.5177, where, the model **(AP1)** and model **(AP3)** select values of N = (1, 3, 1.234153) for all $w_i \geq 0$, i = 1, 2, 3. On the other hand, model **(AP2)** uses values of the first and third proposed normal models = (1, 3, 0) for weights of objectives, which are (0.10, 0.30, 0.60), and weights of constraints are (0.001, 0.592557821, 0.406442179). Wheras, when the values of N is selected as (1, 3, 1), the weights of objectives are (0.10, 0.30, 0.60) with the weights of constraints are (0.0, 0.087051514, 0.912948486). Otherwise, the weights of objectives are (0.0456906, 0.10, 0.8543094) with the weights of constraints are (0.0, 0.0, 1.0). When N= (1, 3, 1.234153), the weights of objectives are (0.10, 0.2117, 0.6883), and the weights of constraints are (0.0, 0.165522, 0.834478).
The special formulations use the value of **D** = 2.5981when the values of N = (1, 3, 1.2341516) for all $w_i \geq 0$, i = 1, 2, 3 in the model **(AP1-a)** and **(AP3-a)**. But, the model **(AP2-a)** uses the values of N = (1, 3, 1) for some weights of objectives such as (0.0456906, 0.10, 0.8543094) with the weights of constraints are (0.0, 0.0, 1.0).

By using the mentioned methods based on the first and third proposed normal models = (1, 3, 0, 0), the best efficient solution can be obtained as: (0.78235954, 2.78235954) or (0.78235955, 2.78235955) for four objectives together. Its distance from utopia point is 3.02. In addition, model **(AP1)** and model **(AP3)** used three groups of weights in objective function, which are: (0.13, 0.05, 0.04, 0.78), (0, 0, 0, 1), or (0.10, 0.07, 0.01, 0.82). They selected values of vector **N** as: (1, 3, 0.93865851, 0.93865851) or (1, 3, 0.9386585, 0.9386585). But when model **(AP2)** used Alia normal = (1, 3, 0, 0), it selected the weights of objectives = (0.01, 0.02, 0.0756029, 0.8943971) and weights of constraint = (0.01, 0.11, 0.13710680, 0.76289320). however, based on the selected values of vector **N** as: (1, 3, 0.3527264, 0.3527264), the weights of constraint are (0.01, 0.11, 0.1118723514, 0.7681276486). The same results are given by the special formulations. It is evident that the best solution from all proposed models is obtained for some weights of objectives because the third and fourth objectives are nonconflicting.

At last, the results show that these proposed methods are more acceptable than previous approach.

Example: 3

The following problem has been presented by Abbas and Huda (Al-Bayati, 2012).

Min: $f_1(x) = x_1 x_5$,

Min: $f_2(x) = x_1^{-1} x_3^2 x_4^4$,

Subject to:

$5x_1^{-1} x_2 \leq 1$,

$2.5x_2^{-1} x_3^2 + 1.5x_3^{-1} x_4^{-0.5} x_5^{-0.5} \leq 1$,

$x_i \geq 0$, where i = 1, 2, 3, 4, 5.

The pay-off matrix is: $\begin{pmatrix} 0.1301946E-06 & 5904.271 \\ 1.21E+35 & 0.2602306E-07 \end{pmatrix}$,

The selected normal of objectives is n_1= 0.8, n_2 = 0.4; w_1 = 0.43689, w_2 = 0.56311, and a = 0.3359.
Then, the optimal solution of model **(AP3)** is x^* = (4.778309, 0.9556618, 0.3228839, 5.370617, 7.597504). But, the special formulation **(AP3-a)** uses n_1= 0.799942, n_2 = 0.399942, **D**= 42.985 with equal weights. Its result is x^* = (13.25349, 2.650698, 0.5377426, 5.370539, 2.739184). The image in the objective space is $F(x^*)$ = (36.3032, 18.1516). Moreover, the value of first constraint= the value of second constraint = 1.0. The distance from the utopian point is 40.5882. In generic, these results are achieved as the previous work by Alia (Gebreel, 2022) but with different values of variables in less time.

6. Conclussion and Future Work

To conclude, it is important to study the existence of advanced Alia's methods to solve the multi-objective optimization problems in general. This work firstly finds the individual optimal of objectives. Then it constructs a plane passing through these individual optimal points, and search orthogonal to the plane. The flexibility of the structure can help to obtain the best compromise solution easily. The experiment results have demonstrated that the proposed methods can provide a better solution in comparison with the previous works. It is also important to mention that the improved methods can be suitable and powerful classical optimization techniques used to solve the two classes of convex and non-convex optimization programming problems. Consequently, these reliable methods are useful for decision-makers to deal with such problems. They are implemented using LINGO software.

For future work, this research will serve as a base for future studies to get the best solution quickly. Additionally, it is possible to apply these proposed methods successfully in more complex problems of the real-world.

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