Exploring Interest Rate Models: Implications on Bond Value Measures in a Dynamic Financial Landscape

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Abstract

This paper investigates the impact of various one-factor no-arbitrage interest rate models on key bond value measures, specifically effective duration. Employing numerical methods based on binomial or trinomial lattices, the study assesses five prominent interest rate models: Ho and Lee, Kalotay, Williams, and Fabozzi, Black, Derman, and Toy, Hull and White, and Black and Karasinski. The analysis begins by outlining the theoretical foundations and assumptions underlying each model, highlighting their distinctive features and implications for bond valuation. Through a meticulous numerical solution process, the study generates risk metrics for bond portfolios, considering the dynamic nature of interest rates and the complex interactions between price, duration, and convexity. Comparisons across the models reveal nuanced differences in the computed effective duration and convexity measures, shedding light on how the choice of an interest rate model may influence risk assessments in fixed-income portfolios. The paper discusses practical implications for investors and portfolio managers, emphasizing the importance of model selection in navigating the challenges posed by interest rate fluctuations. Additionally, it addresses the potential limitations and challenges associated with each model, offering insights into their relative strengths and weaknesses. By presenting empirical examples and conducting sensitivity analyses, this research contributes to the ongoing discourse on interest rate modeling and its implications for bond markets. The findings offer valuable insights for practitioners seeking to enhance their risk management strategies in fixed-income investments, providing a foundation for future research in this dynamic and evolving field.

Keywords: Agricultural Insurance, Black-Scholes Method, Rainfall Index.

1. Introduction

Interest rates play a crucial role in the valuation of bonds. The relationship between interest rates and bond values is fundamental and is based on the time value of money. Bonds typically pay periodic interest, known as coupon payments, and return the principal at maturity. The future cash flows from a bond are discounted back to present value using the prevailing interest rate. Consequently, higher interest rates lead to lower present values, directly impacting the current market price of the bond (Lioui, 2001).

Yield to Maturity (YTM) is a key metric in bond valuation. It represents the total return anticipated on a bond if it is held until it matures. YTM is the discount rate that equates the present value of a bond's future cash flows to its current market price. Changes in interest rates directly affect the YTM, influencing the bond's market price (Weniasti, 2019).

Bonds exhibit interest rate risk, meaning their values are sensitive to changes in interest rates. When market interest rates rise, the value of existing bonds typically falls because their fixed coupon payments become less attractive compared to newly issued bonds at higher rates. Conversely, when interest rates decline, existing bonds may experience an increase in value (Viceira, 2012).

Duration and convexity are measures that help investors understand and manage interest rate risk associated with bonds. Duration quantifies a bond's sensitivity to interest rate changes, while convexity captures the curvature in the relationship between bond prices and interest rates (Shaffer, 2007). These metrics assist investors in assessing and mitigating the impact of interest rate fluctuations on their bond portfolios.

No-arbitrage models typically initiate with identical or comparable Stochastic Differential Equation (SDE) models but leverage market prices to construct an interest rate lattice. This lattice, in turn, generates a spectrum of interest rates that yield bond prices mirroring those observed in the market and adhering to the behavioral characteristics outlined in the SDEs. Several renowned models adopting this methodology include Ho and Lee (1986), Black,

Derman, and Toy (1990), Hull and White (1990, 1993), Black and Karasinski (1991), and Heath, Jarrow, and Morton (1992).

The favored framework for valuing interest rate derivatives is the no-arbitrage model. This preference arises from its capability to accurately determine market prices for bonds. In contrast, equilibrium models fall short of providing precise bond pricing, potentially leading to notable impacts on associated contingent claims.

We examine differences in the effective duration from different one-factor no-arbitrage interest rate models. The models considered are: the Ho and Lee (HL) [1986] model; the Kalotay, Williams, and Fabozzi (KWF) (1993) model; the Black, Derman, and Toy (BDT) (1990) model; the Hull and White (HW) (1994) model; and the Black and Karasinski (BK) (1991) model.

2. Literature Review

2.1. No Arbitrage Interest Rate Models

The short-term rate of interest is assumed to follow a specific process in the interest rate models we look at, which can be modeled by a stochastic differential equation. Every interest rate model is an instance of the general manner that short-term rate fluctuations occur:

$$df\big(r(t)\big) = \Big(\theta(t) + \rho(t)g\big(r(t)\big)\Big)dt + \sigma(r(t), t)dz \tag{1}$$

Where ρ is the mean reversion term to an equilibrium short-term rate, θ will be demonstrated to be the drift of the short-term rate, and f and g are appropriately selected functions of the short-term rate and are the same for the majority of models shown here. The short-term rate's local volatility is expressed by the term σ, and the unpredictability of any future changes in the short-term rate is represented by the normally distributed Wiener process, z.

The short-term rate (one factor) is the only result of the one-factor model in equation (1) (Black, 1991). The expected or average change in the short-term rate over a brief period of time is its first component, the dt term. Given that it contains the random component dz, the second component is the risk term. Every interest rate model that we take into consideration is an instance of equation (1).

2.1.1. The Ho Lee Model

According to the Ho-Lee model, $f(r) = r$ and $\rho = 0$ can be used to model changes in the short-term rate using Equation (1). This means that the short-term rate process is as follows:

$$dr = \theta(t)dt + \sigma(t)dz \tag{2}$$

The HL process is a normal process for the short-term rate since dz is a normally distributed Wiener process. Equation (2) illustrates that if the random term is big enough to outweigh the drift term (dt), the short-term rate could go negative. This is a significant flaw in the HL model, despite the argument that it doesn't matter if some of its assumptions are erroneous as long as the model produces favorable prices for bonds with embedded options. However, the fact that the short-term rate's volatility is independent of its level and does not imply a return to a long-term equilibrium rate represents yet another potential flaw in the model.

In the other models, some of these restrictive presumptions are liberated. It should be noted that the values of the embedded contingent claim will likely be biased by the distributional properties.

The HL model, which is a very popular interest rate model, is simple to use and provides reasonable prices in a variety of scenarios.

2.1.2. The Kalotay-Williams-Fabozzi Model

According to the Kalotay-Williams-Fabozzi model, $f(r) = \ln(r)$ (where ln is the natural logarithm) and $\rho = 0$ can be used to model changes in the short-term rate using Equation (1). The short-term rate process is obtained by modifying Equation (1) as follows:

$$d\ln(r) = \theta(t)dt + \sigma(t)dz \tag{3}$$

Equation (3) and Equation (2) show that the KWF model and the HL model are exactly the same, with the exception that the KWF model now models the change in the short-term rate's natural logarithm rather than the change in the short-term rate itself. Given that r itself has a lognormal process and that $\ln(r)$ has a normal process, the KWF model is a lognormal interest rate model. Therefore, since $r = e^{\ln(r)}$, r itself will never be negative even though $\ln(r)$ may become negative if the risk component in Equation (3) dominates the drift component. As a result, the KWF model solves the HL model's potential issue with negative short-term rates.

The drift term is not included explicitly in the KWF model itself. It consequently doesn't always have a binomial tree solution. Even though the KWF model can prevent short-term rates from going negative, mean reversion in the short-term rate is still not captured by it.

2.1.3. The Black Derman-Toy Model

The Black-Derman Toy model's ability to represent a realistic term structure of interest rate volatilities is one of its key advantages. It is a lognormal model. The drift in interest rate movements is dependent on the level of rates in order to achieve this feature, which allows the short-term rate volatility to fluctuate over time.

This feature is introduced by the term structure of volatilities, even though interest rate mean reversion is not explicitly modeled. This means that the local volatility process determines how much the drift term depends on the rate level. As a result, the mean reversion term does not need an extra degree of freedom, and the binomial tree approach can be used to approximate the BDT SDE rather easily.

Equation (1) can be used to create the BDT model by setting $f(r) = \ln(r)$ and $g(r) = \ln(r)$. Consequently, the lognormal process is followed by the short-term rate in the BDT model:

$$d\ln(r) = (\theta(t) + \rho(t)\ln(r))dt + \sigma(t)dz \qquad (4)$$

The local volatility of the interest rate affects the mean reversion term ρ(t) in the following ways:

$$\rho(t) = \frac{d}{dt}\ln(\sigma(t)) = \frac{\sigma'(t)}{\sigma(t)}$$

Which gives

$$d\ln(r) = \left(\theta(t) + \frac{\sigma'(t)}{\sigma(t)}\ln(r)\right)dt + \sigma(t)dz \qquad (5)$$

When we compare Equation (5) to Equation (3), we can see that the BDT model reduces to the KWF model if the volatility term structure is flat and $\sigma(t)$ is constant. This is because $\sigma'(t) = 0$ and $\rho(t) = 0$. The BDT for constant local volatility is thus a special case of the KWF model. The BDT model will show mean reversion if the local volatility term structure is decreasing, that is, if $\sigma'(t) < 0$. In case $\sigma'(t) > 0$, meaning that the local volatility term structure is growing, mean reversion will not be observed in the BDT model. Therefore, the form of the local volatility term structure determines the mean reversion in its entirety.

2.1.4. The Hull-White Model

The Hull-White model makes the same assumption about the short-term rate as the Ho-Lee model. Equation (1) can be used to obtain the model by setting $\rho = -\varphi$ and $f(r) = g(r) = r$. The short-term rate procedure is as follows:

$$dr = (\theta - \phi r)dt + \sigma dz \qquad (6)$$

Where the mean reversion term is denoted by φ and the long-term equilibrium mean rate is represented by θ. Take note that the HW process reduces to the HL process if $\varphi = 0$. Therefore, in the absence of mean reversion, the HL model is a particular instance of the HW model.

By identifying a central tendency for the short-term interest rate and defining the rate at which the rate reverts to this central tendency, the Hull and White (HW) model explicitly incorporates mean reversion. By minimizing unchecked growth or decline in the model, this mean reversion coefficient lessens the possibility of negative interest rates. It does not, however, completely rule out the possibility of negative rates, leaving the HW model vulnerable to criticism in line with that of the Ho and Lee (HL) model.

Using a binomial tree to solve the related stochastic differential equation (SDE) numerically becomes more difficult when mean reversion in the HW model is explicitly modeled. An extra degree of freedom is required, which leads to this complexity. The analysis becomes more complex when using time steps of different lengths in a binomial framework to obtain this additional degree of freedom. The explicit mean reversion in the HW model can be addressed more manageably by employing a trinomial lattice as an alternative method for tackling the numerical solution.

2.1.5. The Black Krasinski Model

We set $f(r) = ln(r)$, $\rho = -\varphi$, and $g(r) = ln(r)$ in Equation (1) to produce the Black-Karasinski short-term rate process, which yields the following short-term rate process:

$$d\ln(r) = (\theta - \phi\ln(r))dt + \sigma dz \qquad (7)$$

Upon closer examination, it becomes clear that the BK model is just the HW model's logarithmic equivalent. The properties of $\ln(r)$ in the BK model are identical to those of r in the HW model. However, because $r = e^{\ln(r)}$ is

always positive, r cannot become negative, just like in the KWF model. This is where the BK model outperforms the HW model.

So, just as the HW process is an extension of the HL process, so too is the BK model an extension of the KWF process. In actuality, the KWF model is obtained as $\varphi = 0$.

By identifying a central tendency for the short-term rate and the rate at which it reverts to that central tendency, the BK model explicitly models mean reversion. The simplest method for solving the BK SDE numerically is to use a trinomial tree approach, just like the HW model, which also has a mean reversion term.

3. Methods

3.1. Numerical Solution of Interest Rate Models

Binomial or trinomial methods can be used to numerically solve the models.

3.1.1. Binomial SDE Approximations

The binomial method adopts a geometrically analogous approach to model the short-term interest rate, similar to equities in Cox, Ross, and Rubinstein [1979]. For the short-term rate in the following period, represented by the symbols r_u or r_i where $r_u > r_i$, there are two possible values for this method. A binomial tree is produced by iterating through several future time periods; each time step adds two nodes to the tree, resulting in a rapid increase in node count.

The algorithm is subject to a constraint called the recombination condition in order to improve computational tractability. This requirement makes sure that the future interest rate from an up move followed by a down move and a down move followed by an up move are equal. This recombination condition limits the number of nodes that can increase at each time step of the binomial method to one, making it more computationally manageable.

Since there is a q-fold chance of an increase in the short-term interest rate, there is a 1-q-fold chance of a corresponding decrease. We make use of q = 0.5. It should be mentioned that the up and down probabilities being set to 0.5 is only a manufactured tool to guarantee risk-neutrality when solving the short-term rate SDEs. It does not, however, imply that there is an actual probability of 0.5 for an increase or decrease in interest rates (Brennan, 1979).

3.1.2. Trinomial SDE Approximations

The main conceptual difference between the trinomial method and the binomial tree is that the former has two possible states, whereas the latter has three: an up move, a down move, and a middle move. Ensuring the interest rate lattice demonstrates the recombination property preserves computational tractability. This property guarantees that certain moves, like an up move followed by a down move, which equals two consecutive middle moves, or a down move followed by an up move, result in the same node. As a result, the number of nodes in the trinomial lattice increases by just two nodes every time step.

The Hull and White method (HW version) is used to solve the Black and Karasinski (BK) model as well as the Hull and White (HW) model to find the probabilities of an up move, middle move, and down move. The condition q1 $q_1 + q_2 + q_3 = 1$ is satisfied by the probabilities for an up move (q_1), middle move (q_2), and down move (q_3) in each version.

For the BK and HW models, probabilities that depend on the mean-reversion term can be included in the trinomial lattices created using the Hull and White approach. Hull and White truncate the upper and lower branches of the lattice at predefined maximum and minimum levels in order to preserve positive probabilities. A new branching procedure is applied with varying probabilities beyond these bounds.

3.2. Effective Duration

One measure that shows how much of a percentage a change in absolute yield has on bond price is called modified duration. Another way to look at it is as the price divided by the negative slope (first derivative) of the price-yield line. Likewise, convexity portrays the second derivative by indicating the curvature of the price-yield relationship.

Nevertheless, neither modified duration nor convexity account for possible variations in cash flows resulting from a bond's embedded option being exercised. They might therefore produce disappointing outcomes for bonds that have embedded options. This limitation is addressed by effective duration, which take into consideration possible changes in cash flows resulting from the exercise of embedded options in response to future interest rate changes.

To offer a more thorough analysis, effective duration (ED) is calculated, recognizing the dynamic nature of cash flows related to embedded options in the face of fluctuating interest rates.

Effective Duration:

$$ED = \frac{P_{down} - P_{up}}{2P_0(\Delta_y)} \tag{8}$$

4. Results and Discussion

We examine callable bonds and putable bonds, which are two bond types that have embedded options, shown in Table 1. Five-year maturities are used in the analysis (Bluetow, 2000). The structures are priced using trinomial and binomial interest rate lattices with six-month time steps. We make use of the volatility and spot rate term structure displayed in Table 2. Lastly, where appropriate, a mean reversion term of 5% is assumed.

A rough representation of the price-yield relationship for each of the two security types is shown in Figure 1. We anticipate that the callable bond will typically have an effective duration of roughly one year at extremely low interest rates.

**Table 1**: Bond Characteristics

	Option Strike	Coupon	Time Option Starts
Callable Bond	$102.50	6.00%	1 year
Putable Bond	$95.00	6.00%	1 year
Callable Range Note	$97.50	Floating	1 year
Putable Range Note	$97.50	floating	1 year

Table 2: Spot Rate and Volatility Term Structures

Year	Spot Rates	Volatility
0.5	6.30	10.00
1.0	6.20	10.00
1.5	6.15	10.00
2.0	6.16	9.00
2.5	6.17	9.00
3.0	6.15	9.00
3.5	6.13	9.00
4.0	6.09	9.00
4.5	6.06	8.00
5.0	6.02	8.00
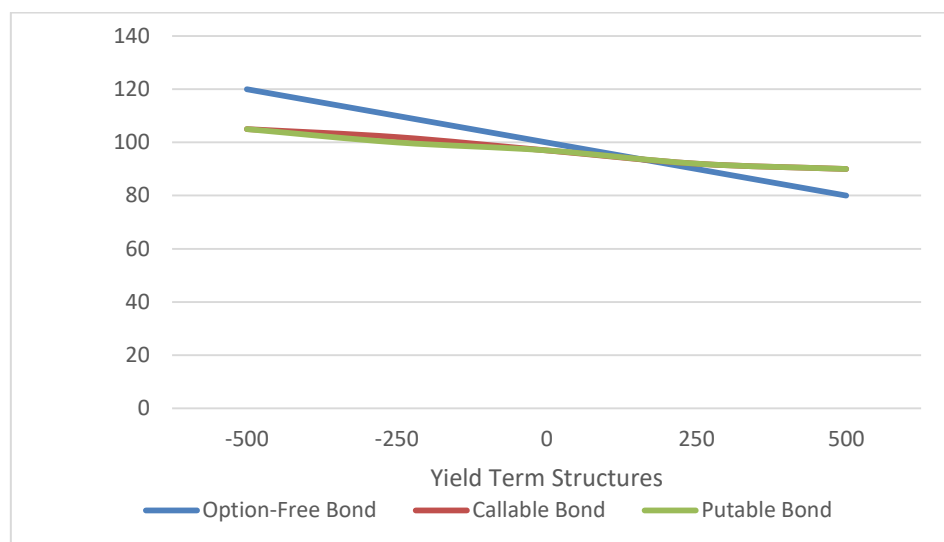
Figure 1: Callable and Putable Bond
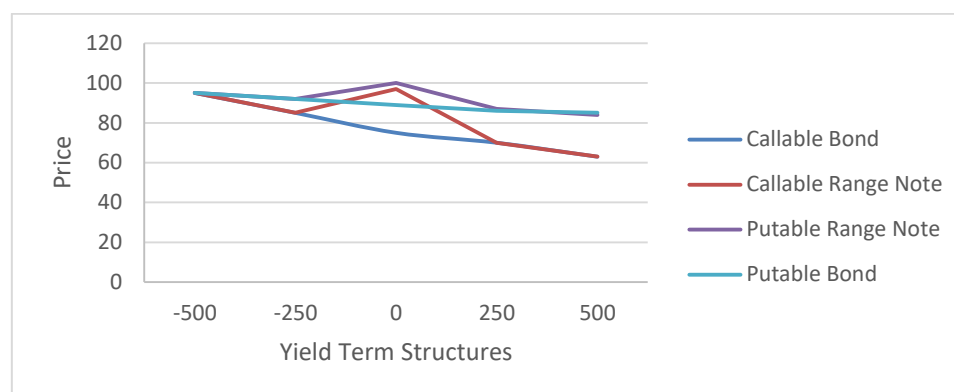
Figure 2: Callable and Putable Range Note

The regular callable and putable bond and the callable and putable range note pricing behavior across a range of yields is depicted in the two graphs. Apart from the standard callable and putable bond, the first graph illustrates the relationship between price and yield for an option-free bond possessing identical attributes. Both the callable and putable zero-coupon bonds' price-yield curves, which have the same features as the matching range notes, are displayed in the second graph. Since these two securities provide a lower bound on the prices of the callable and putable range notes, they are only displayed for reference.

Because the issuer has an incentive to call the issue on the earliest call date and refinance its debt at a lower interest rate, the effective duration should be low. As a result, at low interest rates, the callable bond's price-yield relationship has a slight slope. Since the issuer has no motivation to call the bond issue before maturity, a callable bond is similar to an option-free bond in an environment of high interest rates. Consequently, in an environment with high interest rates, the effective duration is roughly equivalent to the effective duration of a corresponding option-free bond.

The price-yield relationships for callable and option-free bonds have nearly identical slopes at high interest rates, but they diverge at lower interest rates, as shown in Figure 1. On the other hand, in low-interest rate environments, the effective duration of putable bonds is similar to that of option-free bonds because bondholders have little incentive to redeem them early. Putable bonds, on the other hand, have shorter effective durations roughly one year in this instance at high interest rates because bondholders typically redeem them early in favor of higher returns elsewhere.

The relationship is more complex for range notes. The price-yield curves for zero-coupon callable and putable bonds with the same strike prices and callability periods are displayed in Figure 1. In contrast, range notes pay a floating-rate coupon when the interest rate is within predetermined bounds. Both the putable and callable range notes must have a value equal to or greater than the matching zero-coupon callable bond. As a result, range notes' duration pattern differs from zero-coupon bonds' only when interest rates are in the range where a floating-rate coupon is given. The range notes' durations outside of this range closely match the corresponding zero-coupon bonds' durations.

Interest rates generally have an impact on range note behavior, with effects aligning at very high rates and diverging at very low rates. Because of the increased discounting of the redemption value and the possibility of coupons being absent, high rates result in a lower present value. On the other hand, the present value increases at low rates, but callable range notes may not be called due to the issuer's inexpensive financing.

The effect of this dynamic on the effective duration of range notes is substantial. Putable range notes have a low effective duration and callable range notes have a high effective duration at high rates. The effective durations of putable and callable range notes differ inversely at low rates. These trends in the price-yield relationships are shown in Figure 1.

Table 3: Effective Duration (Binomial)-1

	-500 bp	-250 bp	Original (0)	250 bp	500 bp
Ho-Lee					
Callable Bond	2.1243	2.5365	2.4526	2.5544	3.0556
Putable Bond	3.4528	3.2383	2.8964	2.7253	2.4569
Callable Range Note	4.3656	-1.7185	2.2345	4.2559	2.7632
Putable Range Note	4.4676	-0.0342	1.9460	1.9601	0.6527
Kalotay-Williams-Fabozzi					
Callable Bond	0.9802	0.9680	3.6940	4.1764	4.0875
Putable Bond	4.4452	4.3552	4.1643	1.4121	0.9322
Callable Range Note	4.4769	-7.3821	1.9101	12.1257	6.0632
Putable Range Note	4.2303	-3.8566	0.5662	3.333	0.9470

Black-Derman-Toy

Callable Bond	0.9802	0.9680	3.8620	4.1765	4.0875
Putable Bond	4.4453	4.3552	4.2656	1.2215	0.9322
Callable Range Note	4.4768	-7.5520	1.1107	13.9608	5.7683
Putable Range Note	4.2419	-3.5625	0.5573	3.6847	0.9470

Table 4: Effective Duration (Binomial)-2

	-500 bp	-250 bp	Original (0)	250 bp	500 bp
Ho-Lee					
Callable Bond	2.5483	2.4641	2.4526	2.5544	3.3456
Putable Bond	3.5943	3.3607	3.2409	2.9184	2.3306
Callable Range Note	1.8843	-3.0793	5.4636	3.3510	3.0201
Putable Range Note	2.9997	-0.9756	2.9736	1.7582	1.5931

Table 3: Effective Duration (BK Trinomial)

	-500 bp	-250 bp	Original (0)	250 bp	500 bp
Black-Krasinski (HW Version)					
Callable Bond	0.9802	0.9681	3.7210	4.1455	4.0584
Putable Bond	4.4809	4.3207	4.1841	1.2907	0.9322
Callable Range Note	4.9290	-8.4850	1.1348	11.0725	5.6202
Putable Range Note	4.2399	-4.5037	0.3010	2.1743	0.9471

The imposition of boundary conditions within interest rate ranges is the reason behind the non-monotonic price-yield relationship of range notes. When rates rise within the range, the coupon effect takes over, increasing the value of the range note and producing an effective duration that is negatively sloped.

The HL, KWF, and BDT models' effective duration results are shown in Figure 2. The KWF and BDT models yield estimates that are comparable, but the HL model's normal interest rate modeling causes it to occasionally deviate, particularly in extreme interest rate scenarios. For example, the effective duration estimate for a callable bond in the HL model is substantially different from lognormal models in very low-interest environments, and the same is true for a putable bond in very high-interest scenarios.

The findings in table 3 are explained by our HL model's lack of truncation at zero. Because of the hyperbolic nature of value with respect to the discount rate, the HL model, which has multiple negative rates, produces higher effective duration (ED) values at very low interest rates. The trinomial versions of the BK and HW models, which both explicitly include mean reversion, are shown in Figures 4 and 5. Compared to the BK model, the HW model—a normal interest rate model—shows less variation in ED estimates with interest rates. Significant variations in duration estimates can be seen in the comparison between Figures 4 and 5, particularly for complex securities such as range notes.

5. Conclusion

The study demonstrates substantial variations in effective duration estimates across different bond structures specifically, regular callable and putable bonds, as well as callable and putable range notes. This variability is evident not only for regular callable and putable bonds but is even more pronounced for intricate bond structures like range notes.

The findings underscore the necessity for a meticulous interpretation of these metrics. The complexity of structures, such as structured notes, collateralized mortgage obligations, asset-backed securities, among others, amplifies the challenges associated with differences in metrics. While no model or implementation framework is deemed superior, users must ensure a comprehensive understanding of the model employed before applying these metrics in an investment or risk management context.

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