A Bibliometric Analysis for Lebesgue Measure Integration in Optimization

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Abstract

In solving mathematical problems so far, Riemann's integral theory is quite adequate for solving pure mathematics and applications problems. But not all problems can be solved using this integration, such as a discontinuous function that is not Riemann's integration. Lebesgue integral is an integration concept based on measure and can solve finite and unlimited function problems and be solved in a more general set domain. One of the bases of this integration is the Lebesgues measure includes the set of real numbers, where the length of the interval is the endpoints. The alternative use of this integral is widely used in various studies such as partial differential equations, quantum mechanics, and probabilistic analysis, requiring the integration of arbitrary set functions. This paper will show a comprehensive bibliometric survey of peer-reviewed articles referring to Lebesgue measure in integration. Search results are obtained 832 papers in the google scholar database and 997 papers using Lebesgue measure integration in optimization. It can also be seen that the research have 4 clusters and 3 clusters respectively with scattered keywords for each cluster. Finally, using bibliographic data can be obtained Lebesgues measure in integration and optimization supports many of the research and provides productive citations to citing the study.

Keywords: Discontinuous Function, Measure Theory, Network Visualization, Lebesgue Integration, Optimization

1. Introduction

In today's development, science is so prevalent in the world, through increasingly sophisticated technology. Humans can easily capture information from the world's internet by relying on gadgets such as cellphones and laptops of the same kind and using them as a theoretical study in understanding the topic of the material that will be studied, such as a bibliography. Someone often becomes the author of a scientific paper that contributes to scientific matters and frequent references to the bibliography. However, until now, the need for bibliometric studies has not been explored by some people. It 'get to better than deep to be a discussion that pullers to the knowledge library. Bibliometric studies can look at the main theoretical and empirical orientations (Liu & Avello, 2020). Bibliometrics is a scientific discipline in literature (Kou et al., 2021), which applies it can limit mathematics and statistics to communication media and other books. The bibliometric concept is a dining table on scientific products and supports evaluating tools and parameters (Lungu et al., 2020). The bibliometric approach always applies the latest scientific discipline by reviewing scholars, cited publications, journals, and countries. Usually, using bibliometric can also provide collaboration results. Several bibliometric uses aim to map the state of written works based on scientific knowledge on a target topic and analyze authors, journals, institutions, articles, and countries with keyword searches. The number of citations and identify new directions in scientific research topics (Mohadab et al., 2020). The bibliometric aim is to review the extensive literature data (Rejeb et al., 2020). Bibliometrics are used as a research tool in the scientific field with a complicated literature review method (Chàfer et al., 2020).

Most people know mathematics only by calculating and wrestling with numbers, even though many benefits can be attributed to economics if you look closely. For example, they were estimating income or curve fluctuations in income with the theory of Black-Scholes. In biology can determining genotypes that are superior to plant breeds. For physics determines the force's results on an object, and for computers program, use python's script to produce an error value. One of the fields of mathematics is analysis, such as integrals. Integration is the basis of mathematical

techniques for calculating quantities such as surface area, substantial volume, curve length, distribution mean, solutions of differential equations, Fourier transform functions denoted by $\int f(x)dx$ for integral f (Jacobs & Westerbaan, 2015). In 1916, Herman Weyl's first appeared the Riemann integration, which put forward integrals' theory using partitions and functions that not integrable in Riemann values 0 and 1. So their functions are limited (Kiria & Pantsulaia, 2016). It should be noted that the Riemann integration is the part covered by the integral Lebesgue(Zhang et al., 2011) so that the Lebesgue integration appears to calculate a measure theory approach that is carried out on finite functions. Now we can define the Lebesgue integral using the algebraic value ω for the symbol of measure (Jacobs & Westerbaan, 2015).

Researchers in their research widely use Lebesgue integration. Search algorithms to solve robust convex optimization (RCO) for distributed nonlinear uncertainty. One of the Scenario Approaches for RCO uses the Lebesgue measure approach (You et al., 2019). This can also be done in further research to deal with robust optimization to deal with the problem of dataset uncertainty in the optimization criteria (Chaerani et al., 2017), and robust linear optimization (RLO), which discusses parametric uncertainty for optimization problems in it (Chaerani & Roos, 2013), then the Lebesgue measure can be recommended as an alternative solution to the problem.

If the Lebesgue integral on the set measure A is denoted as $\int f(x)d\lambda$ Lebesgue integral A, with the notation λ as the measure of Lebesgue (Fernández-Sánchez et al., 2020). In this study, we survey the total number of papers in the database using a goggle scholar regarding the Lebesgue measure in integration. It can also be seen that the how research cluster was buid and bibliographic mapping analysis and examine the Lebesgue measure in integration theme that researchers have not directed.

2. Method

This bibliographic survey is a search for literature that has been published about Lebesgue measure and integration (LMI) in peer-reviewed journals published in international languages (written in English). In this study, we limit our focus to only the papers published in 2010-2020. The search literature utilizes the google scholar (GLS) database using Publish or Perish software designed for individual academics to represent and obtain citations from various data sources (Crossref, Google Scholar, Scopus, Microsoft Academic, and Web of Science) (Harzing, 2020). Results of Publish or Perish can be saved in various formats, with a custom format reader. This study used the Research Information Systems (RIS) format for bibliographic citations stored in .ris files.

The literature search was obtained to find all papers that included the search terms "Lebesgue Measure" and "Integration." The search engine does this by searching for these keywords, and it gets a row of Cites, Per year, Rank, Authors, Year, Publication, Publisher, and Type. All papers obtained are sourced from Google Scholar because the data source is open and can be used free of charge with no search limit. However, we limit the maximum search to 1000 articles.

The search results obtained are "Lebesgue Measure" and "Integration" with 832 papers; keyword search is only applied to titles and abstracts. The results using Publish or Perish, then VOSViewer software is used to map and visualize and analyze trends for bibliometric mapping. (Van Eck & Waltman, 2010).

3. Bibliometric Analysis

3.1. Research Publication Summary

This session shows a summary of the papers published in Lebesgue Measure and Integration. The number of papers obtained is, as shown in Figure 1. The graph shows the various publication patterns from 2010-2020, showing that Lebesgue Measure and Integration is still used as research support. The number of published articles varies from year to year.
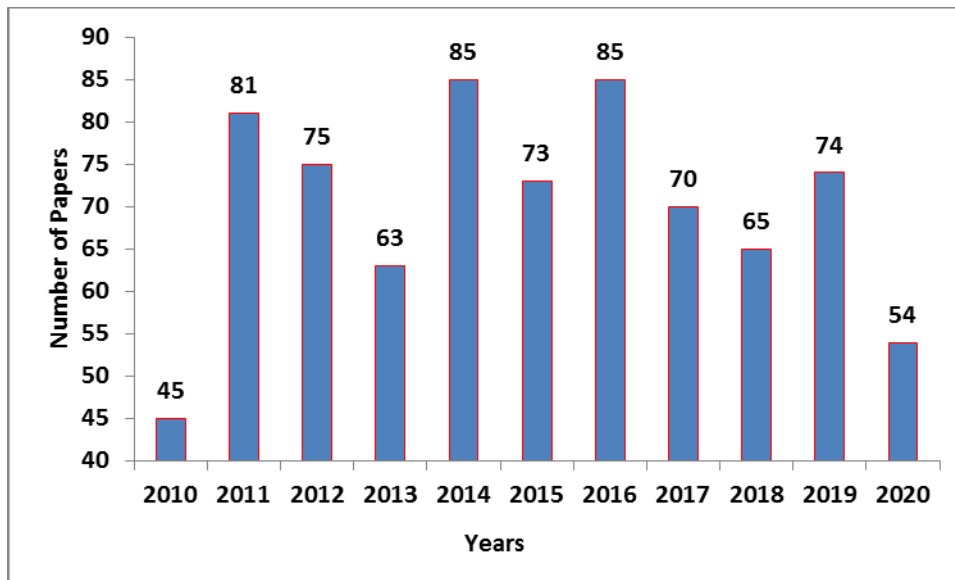
Figure 1: Number of publications of LMI per year

The articles published are in various scientific fields such as pure and applied mathematics, earth science, computer science, electrical, mechanical, information science, statistical probability, etc.

Table 1: Top journals that publish LMI

Journal title	Papers
arXiv preprint	101
Journal of complexity	15
SIAM Journal on numerical analysis	11
Journal of mathematical analysis and applications	10
Measure and integration	9
Journal of functional analysis	8
Bernoulli	6
Electronic journal of probability	5
Real analysis exchange	5
Others	656

Table 2: Top publisher that publishes LMI

Publisher	Papers
Springer	161
arxiv.org	92
Elsevier	80
books.google.com	64
projecteuclid.org	30
Wiley Online Library	16
ieeexplorer.ieee.org	14
SIAM	13
Taylor & Francis	13
Others	340

3.2. Main Research Topic

In the discussion of this section with the theme Lebesgue measure and integration. By using software that can visualize bibliometric networks. The coverage can be obtained, such as journal publications, authors as researchers, and individual publications built on citations either individually or simultaneously (Templeton, 2020). The VOSviewer software can thoroughly perform the task mentioned above to construct and visualize a network of events together from essential terms extracted from scientific literature (Van Eck & Waltman, 2011).

It can be seen in Figure 2 that the co-occurrence network mention LMI literature and identified the clusters in different colors. For example, each cluster keyword correlation shows the co-occurrence of the research network that has been conducted. The more co-occurrence among networks shows that the research is a lot done. On the contrary, if a little connected in the network shows that research with these keywords is an opportunity to become a prospective research topic. As seen in cluster 1 (red), numerical integration and probability measure keywords have a large co-occurrence network, while the keywords in dimensional Lebesgue measure and quasi Montecarlo integration have a small co-occurrence network. It can be used as a reference as a prospective research theme. Whereas in cluster 2 (green), stochastic integration has many co-occurrence networks, but on the other hand, vector measure and fractional integration keywords have a small number. In cluster 3 (blue), integration theory has many co-occurrence networks, behind the number of Riemann. The relationship between clusters allows it to be used as a reference for prospective for the next research.

Table 3:The most cited journal papers on Lebesgue measure in integration

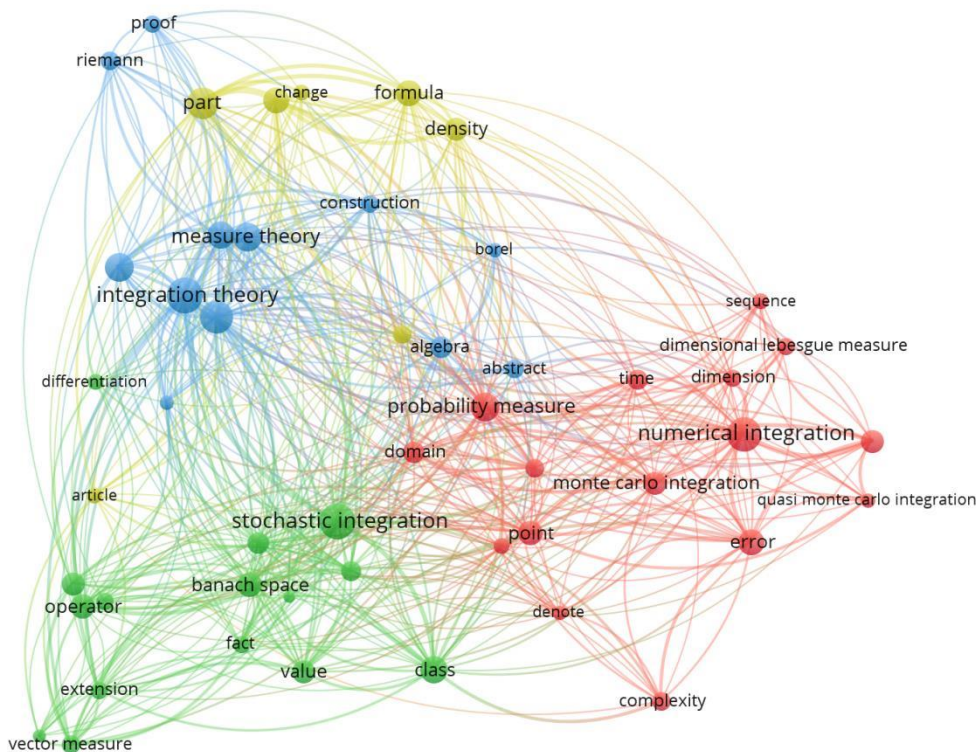| Reference	Cites
SE Mousavi, N Sukumar, 2011, Computational Mechanics, Springer (Mousavi & Sukumar, 2011)	177
CJ Oates, M Girolami, N Chopin, 2014, arXiv preprint, arxiv.org (Oates et al., 2016)	153
M Griebel, M Holtz, 2010, Journal of Complexity, Elsevier (Griebel & Holtz, 2010)	120
G Leobacher, F Pillichshammer, 2014, Elsevier (Leobacher & Pillichshammer, 2014)	115

Figure 2: Bibliographic mapping of LMI research

It can be seen that the network visualization in each colour represents one cluster. Each cluster is grouped in an interrelated circle among items and with other items in other clusters. In the bibliographic mapping of LMI research, there are 83 terms with the most occurrences of numerical integration (63), stochastic integration (68), integration theory (66), part (50) of the four clusters, and there are keywords shown in Table 4.

Table 4: Popular keywords in each cluster of LMI research

Cluster 1	Cluster 2
complexity	banach space
denote	class
dimension	convergence
dimensional Lebesgue measure	differentiation
discrepancy	extension
domain	fact
error	fractional integration
Montecarlo integration	integration operator
numerical integration	keywords
point	measure space
probability	operator
probability measure	real line
quasi Montecarlo integration	stochastic integration
sequence	value
study	vector measure
time	

Cluster 3	Cluster 4
abstract	article
algebra	change
borel	density
chapter	Formula
construction	part
finite measure	parts formula
integration theory	solution
Lebesgue integral	
Lebesgue integration	
measure theory	
proof	
Riemann	

3.3. Most Productive Authors

Apart from bibliographic mapping, VOSviewer can also construct a mapping of co-authorships from previously obtained bibliographic data. A co-authorships map is built from co-occurrence keywords or co-citation mappings. Utilizing Publish or Perish search that has yielded information on RIS files with 832 papers can be used to analyze co-authorships.

Tabel 5: Total link strength of authorship

Author	Documents	Total link strength
dick, j	11	7
pillichshammer, f	7	5
gnewuch, m	7	3
bally, v	7	1
cont, r	5	1
hinrichs, a	5	1
ito, y	9	0
acary, v	7	0
girolami, m	7	0
benth, fe	5	0
könig, h	5	0
stroock, dw	5	0

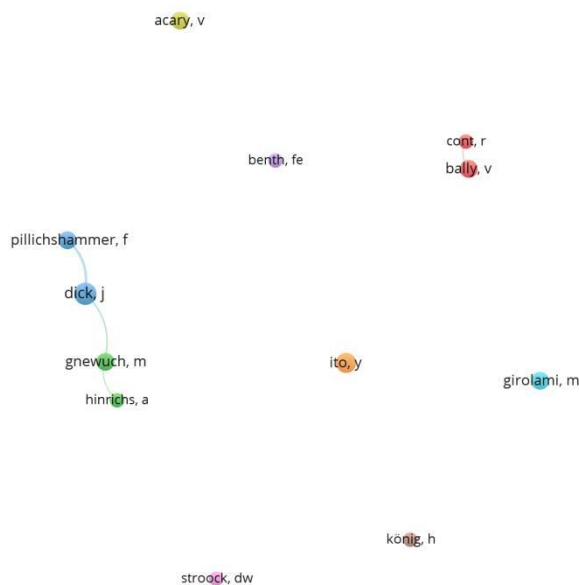**Figure 3**: The largest set of connected co-authorship

Figure 4: Network visualization of 12 selected co-authorship

The type of analysis is co-authorship, and the counting method is full counting is used to create the mappings. The maximum number of authors per document is 25. The threshold used for analysis is the maximum number of documents of an author is 5. From the 1087 authors found in the Publish or Perish search, only 12 meet the threshold, as shown in Figure 4. Next is obtained verify selected authors link with co-authorship to calculate the strength of the selected links. Figure 3 show that the largest set of 12 authors has only four the largest set of connected co-authorship.

3.4. Lebesgue Measure Integration and Optimization

In the optimization section, this study also has an almost close relationship with the Lebesgue measure. Several articles discuss the relationship between the two fields of science. Use of the Google Scholar database index on Publish or Perish for the literature published by Lebesgue Measure Integration in Optimization (LMIO). Then select the title words "Optimization", meanwhile the article keyword and search the abstract "Lebesque measure integration". The results obtained from these keywords are 997 papers. From the search results, it can be seen.

Table 6: Top journals that publish LMIO

Journal title	Papers
arXiv preprint	101
SIAM Journal on Optimization	25
Information sciences	15
Journal of Global Optimization	14
Applied Soft Computing	12
Other	830

Table 7. Top publisher that publishes LMIO

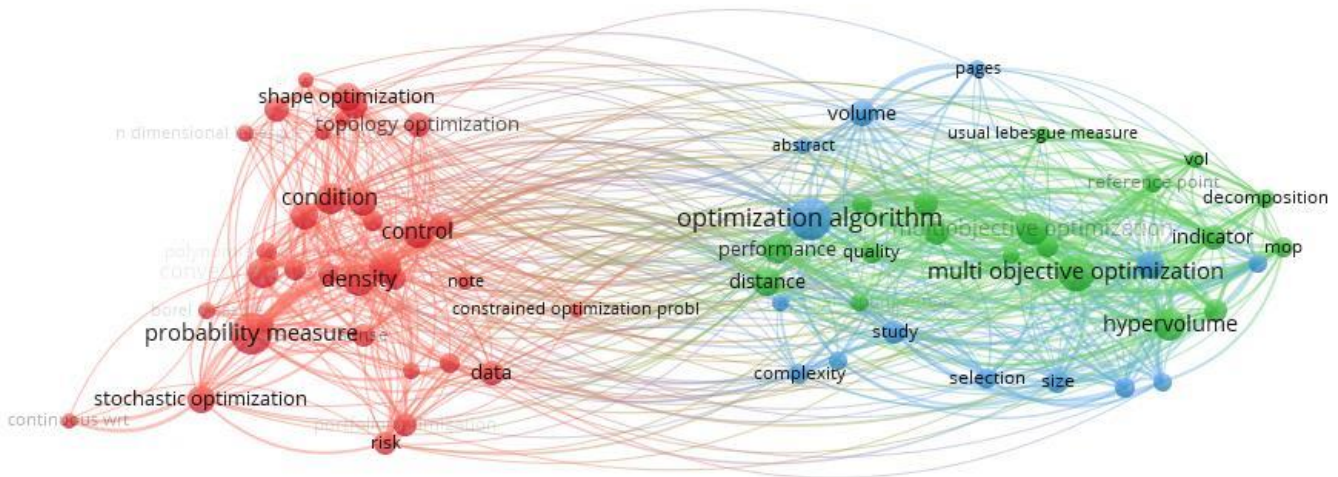| Publisher	Papers
Elsevier	153
ieeexplore.ieee.org	146
arxiv.org	117
SIAM	50
books.google.com	43
Other	488

Figure 5: Bibliographic mapping of LMIO research

In Figure 5 show the bibliographic mapping of LMIO research, there are 115 terms meet the threshold most occurrences of probability measure (75) represent cluster 1 (34 items), optimization algorithm (79) represents cluster 2 (19 items), and multi objective optimization (57) represents cluster 3 (14 items).

4. Discussion

4.1. The latest Result

Many studies on the Lebesgue measure in integration has some papers in the recently published attention. Discussion of the Riemann-Stieltjes integral comparison with the Lebesgue-Stieltjes theory regarding the type of function constant at intervals whose boundary is the point of discontinuity (Niang et al., 2020). Then the discussion on identifying Banach space and functions on limited measurement space to develop some basic integration theory elements (Leinster, 2020). Then, assign to semi-algebraic set categories and the full Lebesgue measurement function and integration theory so that the main results of the classical method's treatment arrangement (Kaiser, 2017). Some examples are for functions that are Lebesgue integrable but not Riemann integrable (Gerver, 2015). The variation of the theory of the relationship between Lebesgue integration and absolute continuity or how to combine the two is an interesting thing in this paper (Bernal, 2018). In multi-objective optimization, the Lebesgue indicator-based evolution algorithm (LIBEA) is introduced to reduce computation costs (Zapotecas-Martínez et al., 2019). Lebesgue sampling-based fault diagnosis and prognosis (LS-FDP) were developed to reduce computational costs and minimize accumulated uncertainty rather than Riemann sampling-based FDP (RS-FDP). Based on recursive least square (RLS) was used to optimize the LS-FDP with experiments on Li-ion batteries (Yan et al., 2016).

4.2. The state of research on Lebesgue measure in integration and optimization

As the bibliographic data presented in Table 4, there are still many opportunities for research and application of the Lebesgue measure in integration according to the topics following Table 4. Keywords that still have little mapping networks can be used as references for new research.

Based on Figure 2, can be considered Figure 6 can be considered bibliometric by keyword fractional integration were recommended for future studies, It can be seen that the network among the terms inside the cluster and outside provides opportunities that are still open for fractional integration research with the Lebesgue measure that can be used as a reference, such as in predictive research using the fractional derivative model and fractional differential equation model (Rusyaman et al., 2020). Linkage of these keywords is an option that can be selected. Whereas in Figure 7, the keyword with dimensional Lebesgue measure, which is strength connected to the integration theory, can provide future research recommendations.
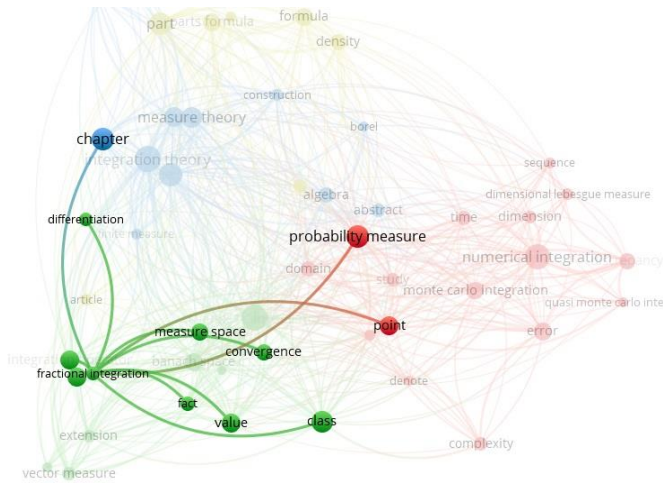
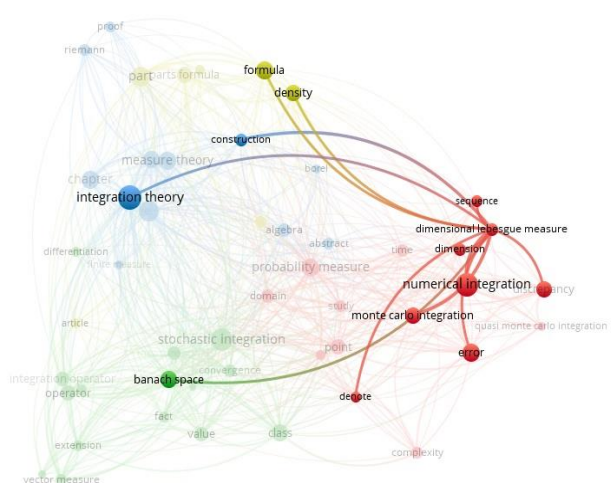**Figure 6**:Bibliometric mapping on fractional integration keyword

Figure 7:Bibliometric mapping on dimensional Lebesgue measure keyword
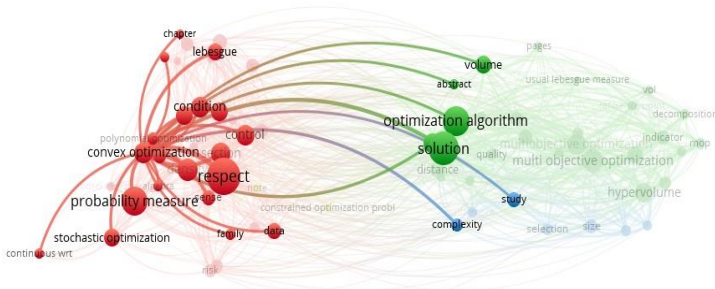

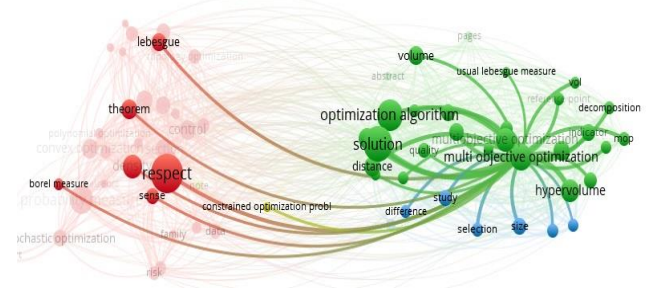**Figure 8**: Bibliometric mapping on convex optimization keyword

Figure 9: Bibliometric mapping on multi objective optimization keyword

With Lebesgue measure integration in optimization, Figure 8 can be seen recommended bibliometric map by keyword convex optimization for the next studies. It appears that the networks among terms are still wide open for convex optimization research with Lebesgue integration. Likewise, there is a correlation between the Lebesgue measure and stochastic optimization, which links keywords that can be selected. Whereas in Figure 9, the keyword with multi objective optimization, which is strength connected to usual Lebesgue measure, can provide next research prospective.

5. Conclusion

The bibliographic analysis survey shows that the Lebesgue measure in integration has a significant role in research, especially regarding integration. This literature research study is quite helpful for further research on Lebesgue integration. The growth in the contribution to this research is wide open and not limited. In this study, bibliometric mapping saw research opportunities on the fractional integration and dimensional Lebesgue measure, which were still open. Meanwhile, the relationship between optimization and Lebesgue measure integration provides a fairly open opportunity and links network visualization strength for further research to stay developed. Using Publish or Perish and VOSviewer software can contribute, although more specific keywords are needed for data retrieval. Thus, the research will continue as the basis for the development of Lebesgue integration.

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